# Modeling Longevity Risks using a Principal Component Approach: A Comparison with Existing Stochastic Mortality Models

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## Abstract

This research proposes a mortality model with an age shift to project future mortality using principal component analysis (PCA). Comparisons of the proposed PCA model with well-known models—the Lee-Carter model, the age-period-cohort model (Renshaw and Haberman, 2006), and the Cairns, Blake, and Dowd model—employ empirical studies of mortality data from six countries, two each from Asia, Europe, and North America. The mortality data come from the human mortality database and span the period 1970–2005. The proposed PCA model produces smaller prediction errors for almost all illustrated countries in its mean absolute percentage error. To demonstrate longevity risk in annuity pricing, we use the proposed PCA model to project future mortality rates and analyze the underestimated ratio of annuity price for whole life annuity and deferred whole life annuity product respectively. The effect of model risk on annuity pricing is also investigated by comparing the results from the proposed PCA model with those from the LC model. The findings can benefit actuaries in their efforts to deal with longevity risk in pricing and valuation.

Keywords: Longevity risk; Age-period-cohort model; Lee-Carter model; Principal component analysis

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## 1. Introduction

Human life expectancy has been increasing significantly since the start of the 20th century, though increments of life expectancy vary with countries and genders. For example, in most Western countries, life expectancies at birth were approximately 66 and 70 years for men and women in 1950, but they increased to 75 and 80 years by 2005 respectively. In Asia, life expectancies were much lower at the turn of 20$^{th}$ century, but the rate of mortality improvement has been much greater, such that life expectancies in Asia today are similar to those of Western countries. For example, life expectancies in 2005 for Japan are 79 and 86 years for men and women; in Taiwan, they are approximately 75 and 81 years (see Figure 1 and Figure 2).

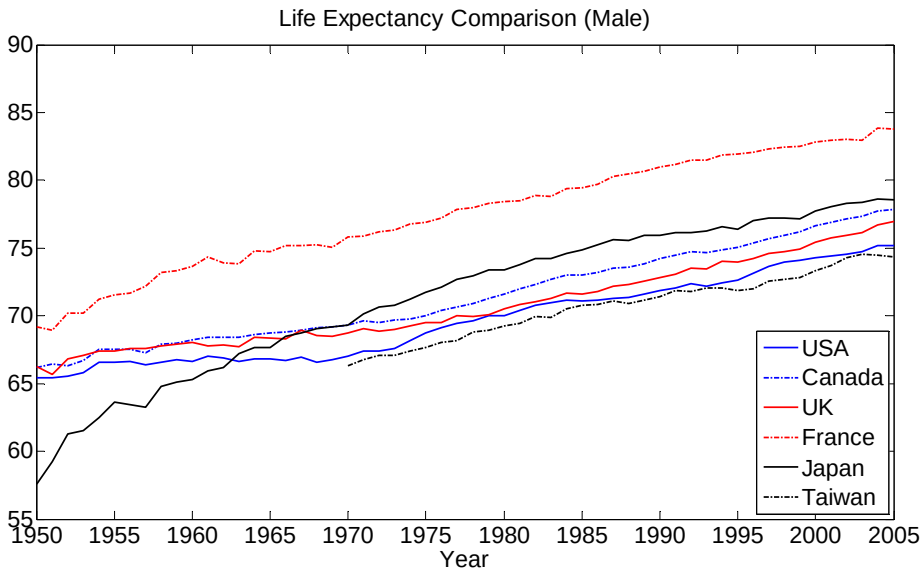



Figure 1. Life Expectancy at Birth for Males in Various Countries:1950–2005[4]

[4] The calculation of life expectancy is based on the period life tables. The mortality data for different countries is accessed from the Human Mortality Database in 2008 except for the country of Taiwan. The data for the country of Taiwan is based on the Minster of Interior, Taiwanese government and data availability limits the Taiwanese data to 1971–2005.

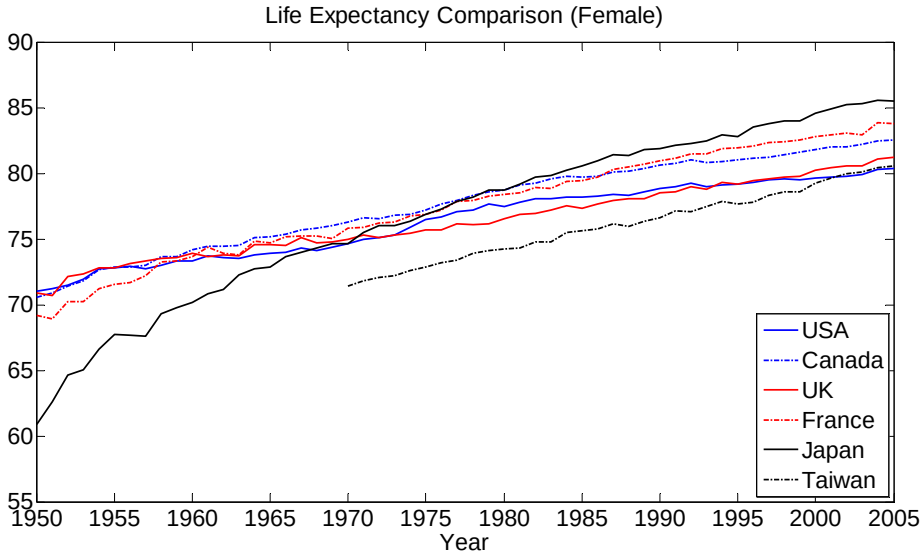



Figure 2. Life Expectancy at Birth for Females in Various Countries:1950–2005

Prolonged life expectancies indicate the possible risk of underestimating premiums by following period mortality tables for life annuity policies. Many previous studies note that mortality risk may cause substantial losses for annuity providers and pension funds, if handled improperly. For example, Antolin (2007) investigates the longevity risk of employer- or provider-defined benefit pension plans in OECD countries. Wilkie et al. (2003) and Ballotta and Haberman (2006) analyze the problem of guaranteed annuity options caused by both interest rate risk and longevity risk. In addition, Bauer and Weber (2007) study the joint impact of investment risk and longevity risk on immediate annuities. To deal with the longevity risk, several scholars suggest the securitization of longevity risk and tackle the valuation methodology by building a mortality index (e.g., Cairns et al., 2006). Such studies use a dynamic mortality model to deal with the mortality risk and note the importance of using a stochastic mortality model for evaluating longevity risk.

In the past two decades, a wide range of mortality models have been proposed and discussed. Among them, the Lee-Carter (1992) or LC model is probably the most popular choice, because it is easy to implement and outperforms other models with respect to its prediction errors (e.g., Koissi et al., 2006; Melnikov and Romaniuk, 2006). Various modifications of the LC model offer broader interpretations (Brouhns et al., 2002; Renshaw and Haberman, 2003), and many countries continue

to use the LC model as the base mortality model for their population projections. For example, the Continuous Mortality Investigation Bureau (CMIB, 2006) in Britain even suggests the LC model as a means to compute stochastic mortality rather than the reduction factor model that it previously proposed.

A good mortality model should resemble historical patterns, and in this sense, the LC model still has room for improvement. For example, the LC model assumes that the logarithms of the age-specific mortality rates are approximately a linear function of time and that the slopes and intercepts in the LC model are functions of ages, which are constant over time. However, many studies show that the parameters are not time-invariant, which would cause larger prediction errors, especially for older age groups (Carter and Prskawetz, 2001; Yue et al., 2008). To fix the parameter problem of the LC model, Cairns et al. (2006) consider a model (CBD model) of functional relationships that deals with mortality rates across ages, which offers better performance for older ages. Renshaw and Haberman (2006) modify the LC model by incorporating a cohort effect. Cairns et al. (2007) further modify the CBD model by adding a cohort effect as well. They reveal that the inclusion of a cohort effect can provide a better fit, and identify the importance of the robustness of parameter estimates over different periods.

In this research, we also propose a model to resolve the problem in the LC model using principal component analysis (PCA), and employ this model to measure longevity risk. We take into account the age shifts in mortality reductions to capture the variant morality improvement at different ages. Similar to Cairns et al. (2007), we compare empirically the proposed model with the LC model, as well as with the age-period-cohort (APC) model proposed by Renshaw and Haberman (2006) and with the CBD model for both model fitting and model forecast. The data for our empirical studies come from Taiwan, Japan (Asia), Britain, France (Europe), the United States, and Canada (North America). We can then use these countries as examples to investigate the pattern of mortality and its modeling for different continents. As performance criteria, we use the mean absolute percentage error (MAPE) and Schwarz-Bayesian criterion (BIC).

In addition to these empirical studies, we use the proposed PCA model to predict future mortality rates and calculate the values of immediate and deferred whole life annuity product separately. With these calculation results, we compare the value of annuity price with that calculated using the period mortality table. Thus, the ratio of annuity price undercharge can be measured. We also investigate the impact of model risk on pricing annuity product numerically. The research findings should benefit actuaries in their efforts to deal with longevity risk in pricing and valuation.

We first summarize the development of mortality modeling in Section 2, followed by a description of the proposed PCA model in Section 3. The empirical studies and their comparisons appear in Section 4, after which we use the proposed model to measure the price of life annuities and discuss model risk in Section 5. Finally, we conclude with discussions and some limitations of our approach.

## 2. Development of Mortality Modeling

In this section, we introduce some popular mortality models and discuss their advantages and limitations. Lee and Carter (1992) propose the following mortality model for the central death rate $m_{x,t}$:

$$\ln\left(m_{x,t}\right) = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}\,, \tag{2.1}$$

where the parameter $\alpha_x$ describes the average age-specific mortality, $\kappa_t$ represents the general mortality level, and the decline in mortality at age $x$ is captured by $\beta_x$. The term $\varepsilon_{x,t}$ denotes the deviation of the model from the observed log-central death rates and is assumed to be white noise with zero mean and relatively small variance (Lee, 2000). The parameter estimates can be derived from matrix operations, such as the singular value decomposition. Equivalently, applying the constraints $\sum_t \kappa_t = 0$ and $\sum_x \beta_x = 1$, the estimate of parameter $\alpha_x$ is the average log-central death

rate over time $t$, such that $\hat{\alpha}_x = \sum_{t=t_1}^{t_1+T-1} \ln(m_{x,t})/T$, where $t_1$ is the starting year and $T$ is the number of years in the data. The parameters $\alpha_x$ and $\beta_x$ are functions of age *x* and do not change with time, and the parameter $\kappa_t$ is a linear function of time. Also, if missing values exist, an approximation method and modifications (Wilmoth, 1996) can be used for the parameter estimation.

The LC model contains relatively few parameters, and it provides fairly good estimates and predictions of the observed mortality rates in many countries, such as the United States and Japan. The LC model thus has gained significant attention since its introduction. According to the assumption of Equation (2.1), mortality improvements at all ages follow a fixed pattern, even though this assumption is unlikely to be true. Usually, younger people experience greater improvements when the mortality starts to decline (e.g., 1960's in Taiwan), and the elderly experience the largest improvements more recently.

Many countries (e.g., Great Britain, Japan) have experienced a similar mortality reduction shift. To investigate the pattern of mortality improvement, Figures 2 and 3 depict the reduction factors[5] for Japanese and British mortality, in which the color represents the reduction factor according to the mortality rates in the base year 1950. Both countries show a similar pattern of improvement; the younger age groups (ages 10 years and below and 20–30 years) experience earlier, largest reductions. Elderly groups experience significant improvements only recently and therefore have the least improvements[6].

[5] The reduction factor can be expressed as the ratio of mortality rate for the same age at different time. In terms of actuarial notion, the reduction factor is calculated as $\frac{q_{x,t}}{q_{x,0}}$, where $q_{x,t}$ is the mortality rate of age $x$ at time $t$.

[6] The reduction factors for other countries are available from the author.

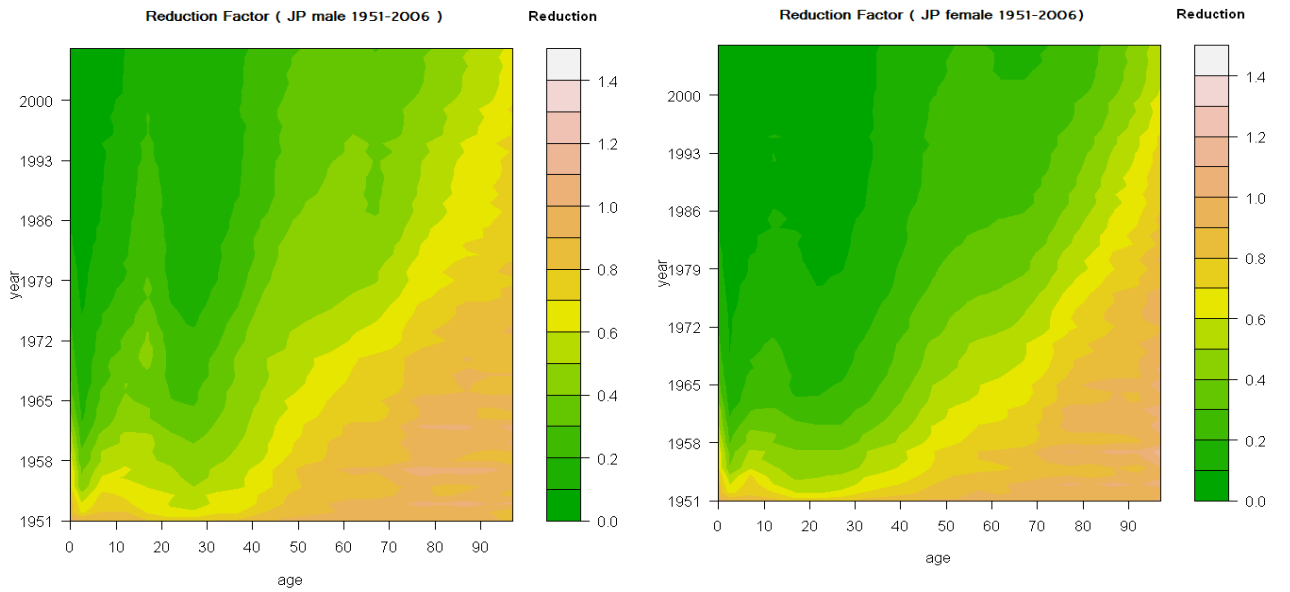



Figure 2. The Pattern of Mortality Improvements of Japanese Men and Women

If we look at the reduction factors closely (Figures 2 and 3), we can find the improvement rates (i.e., slope $\beta_x$ in Equation (2.1)) are not constant with time. In other words, a shift in age occurs for the largest mortality reduction (Booth et al., 2002), which indicates that the assumption in the LC model is not true.

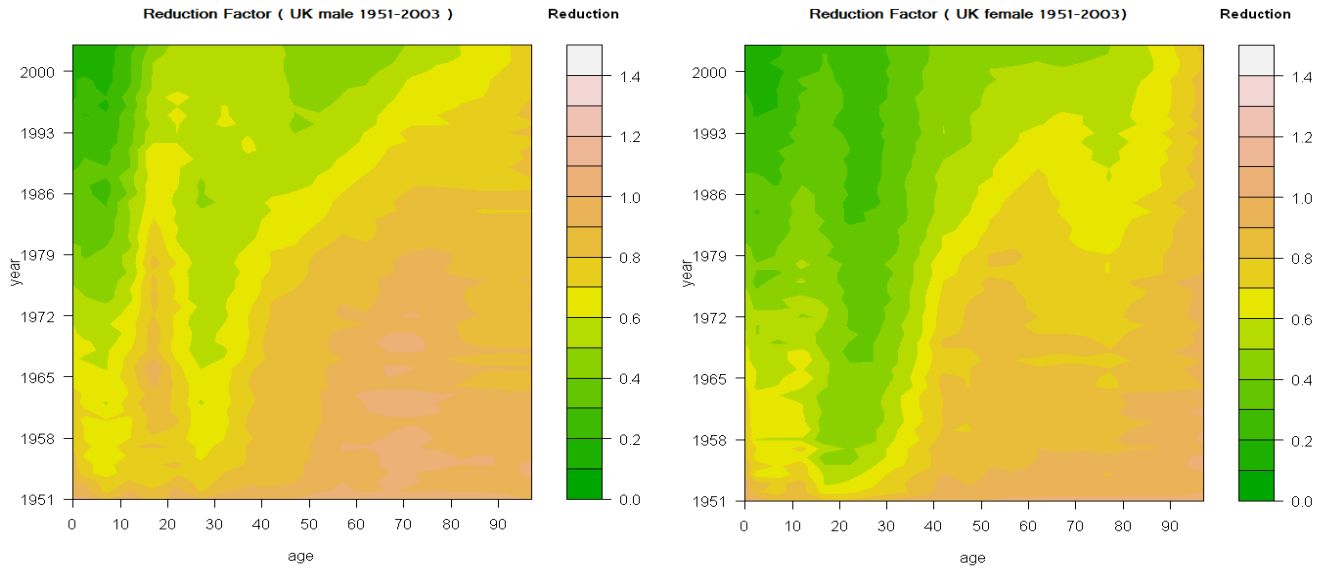



Figure 3. Mortality Improvements of British Men and Women

Another limitation of applying the LC model is the limiting mortality rates of each age. As long as the slope $\beta_x$ is not zero, the linearity of $\kappa_t$ in time implies that the limiting mortality rate is zero for all ages. Several proposed modifications attempt to cope with this limitation. For example, the reduced shift of ages for different time periods can be treated as a "cohort" effect. The original LC model nearly combines the age effect and the interaction of age and time, so possible modifications create additional terms related to the cohort effect. For example, Booth et al. (2002) propose adding more than one interaction term of age and time, such that

$$\ln(m_{x,t}) = \alpha_x + \sum_{j=1}^{J} \beta_x(j)\ \kappa_t(j) + \varepsilon_{x,t}\,, \tag{2.2}$$

where $\beta_x(j)\ \kappa_t(j)$ is the $j$th interaction term between age and time, $j = 1, 2, ....., J$.

Renshaw and Haberman (2003) investigate an LC model with age-specific enhancements for mortality forecasts. Hyndman and Ullah (2005) further suggest using principal component (PC) decomposition to solve for the paired parameters ($\beta_x(j), \kappa_t(j)$). The idea behind this approach is similar to that proposed by Bell (1997), according to which the LC model displays similar behavior for both one and two PCs.

In 2006, the U.K.'s Continuous Mortality Investigation Bureau (CMIB) used Renshaw and Haberman's (2006) proposal to incorporate the cohort based on the LC model, similar to the modification offered by Hyndman and Ullah (2005), such that

$$\ln(\mu_{x,t,c}) = \alpha_x + \beta_x(t)\ \kappa_t + \beta_x(c)\ \kappa_c^* + \varepsilon_{x,t,c}\,, \tag{2.3}$$

where $\mu$ is the force of mortality, and $\kappa_c^*$ is the cohort effect. The model in Equation (2.3) is also known as the age-period-cohort (APC) model built on the LC framework. And it is a special case of the APC model that includes only one main effect (age) and two second-order interaction terms (age-period and age-cohort). The CMIB suggests using a likelihood method for parameter estimations and the classical multivariate time series method for predictions.

Some models instead try to capture the dynamics of older age groups. For example, Cairns et

al. (2006) suggest a two-factor model for modeling initial mortality rates instead of central mortality rate and fit the mortality curve based on the post-age-60 mortality in the United Kingdom. The mortality rate for a person aged x in year t, or $q(t,x)$, is modeled as:

$$\text{logit } q(t,x) = \beta_x^1 k_t^1 + \beta_x^2 k_t^2, \tag{2.4}$$

where the parameter $k_x^1$ represents the marginal effect of times on mortality rates, and the parameter $k_x^2$ portrays the old age effect on mortality rates. This model can be presented in a simple parametric form by setting $\beta_x^1$ equal to 1 and $\beta_x^2 = x - \bar{x}$. Thus, the mortality rate can be modeled as in Equation (2.5):

$$\text{logit } q(t,x) = k_t^1 + k_t^2 (x - \bar{x}), \tag{2.5}$$

where $\bar{x}$ is the mean age.

The CBD model has been widely adopted to investigate issues of hedging and the securitization of longevity risk (Cairns et al., 2006; Wang et al., 2008). In recent years, some extensions of the CBD model incorporate the cohort effect (Cairns et al., 2009）. Cairns et al. (2009) in particular demonstrate that the inclusion of a cohort effect can provide a better fit, using data from England, Wales, and the United States.

## 3. Proposed PCA Model

### 3.1. Principal component Analysis for Mortality Experience

Similar to Bell (1997) and Hyndman and Ullah (2005), we apply the PC approach to the logarithm of central mortality rates. We propose a two-PC model with age shift in the second PC. The LC model can be treated as a one-PC model, and the first PC is a linear function of time. According to Bell (1997), the logarithms of mortality rates can contain one, two, or three PCs, depending on the data. We analyze the PCs for six counties of Japan, Taiwan, Great Britain, France,

Canada, and the United States[7]. To provide readers a basic description of the PCA and its relation with the mortality study, we give some technique notes and interpretation of PCA in Appendix A.

The first two PCs of the logarithm mortality rates for Japanese and British subjects appear in Figures 4 and 5. The left graphs show the first PC, like the linear trend (i.e., $\kappa_t$) in the LC model, and the right graphs are the second PC. The first two PCs account for 98.72 percent (Japanese men) and 99.52 percent (Japanese women) of the variations, and the two-PC models explain approximately 5 percent more variation than the one-PC models. Of course, adding more number of PCs can improve the model fitting but would also increase the risk of adding pure fluctuation. In the case of Japan, the two-PC model outperforms other model such as 1-PC and 3-PC models. The case of British data shows similar results. Therefore, we focus on the two-PC model in this research. We only show the result for Japanese and British data to demonstrate the motivation behind our approach. The PC analysis results for other countries are plotted in Appendix. B.

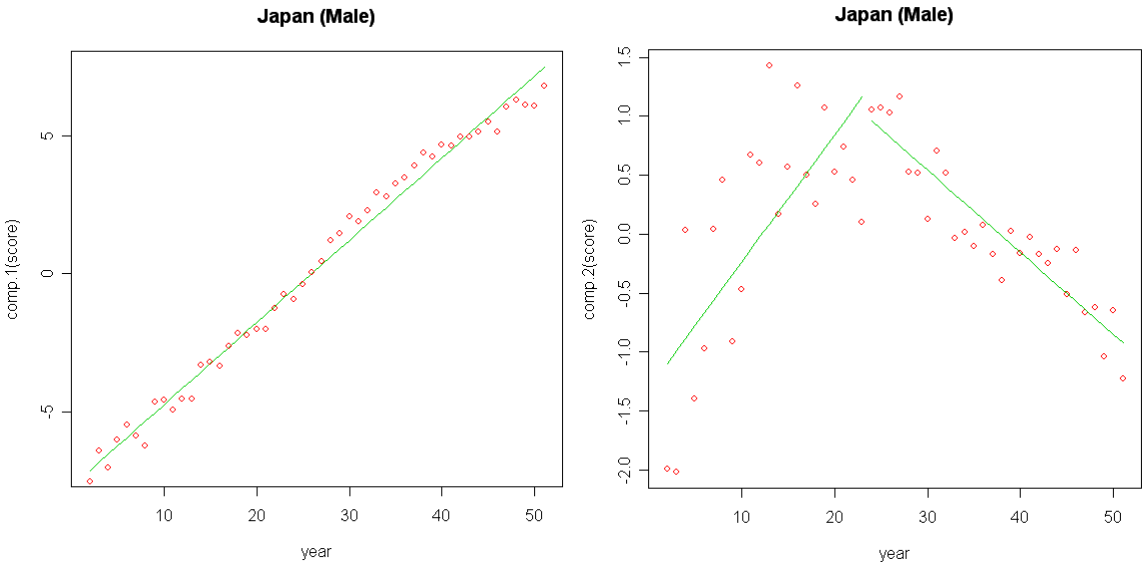



Figure 4. First and Second PC of Japanese Men

[7] See Section 4.1 for the description of mortality data.

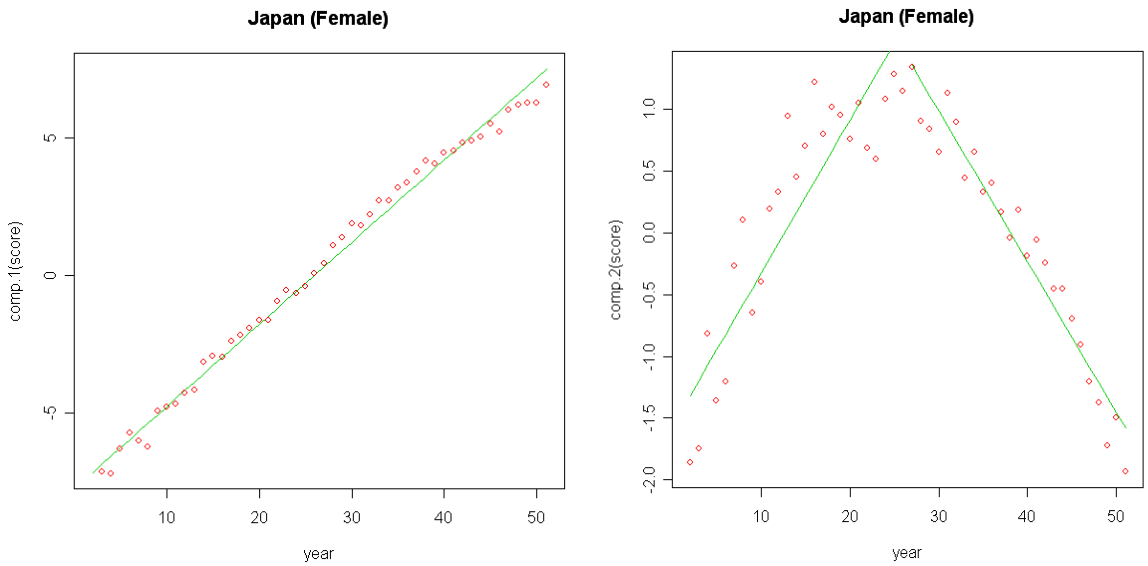



Figure 4(cont.). First and Second PC of Japanese Women

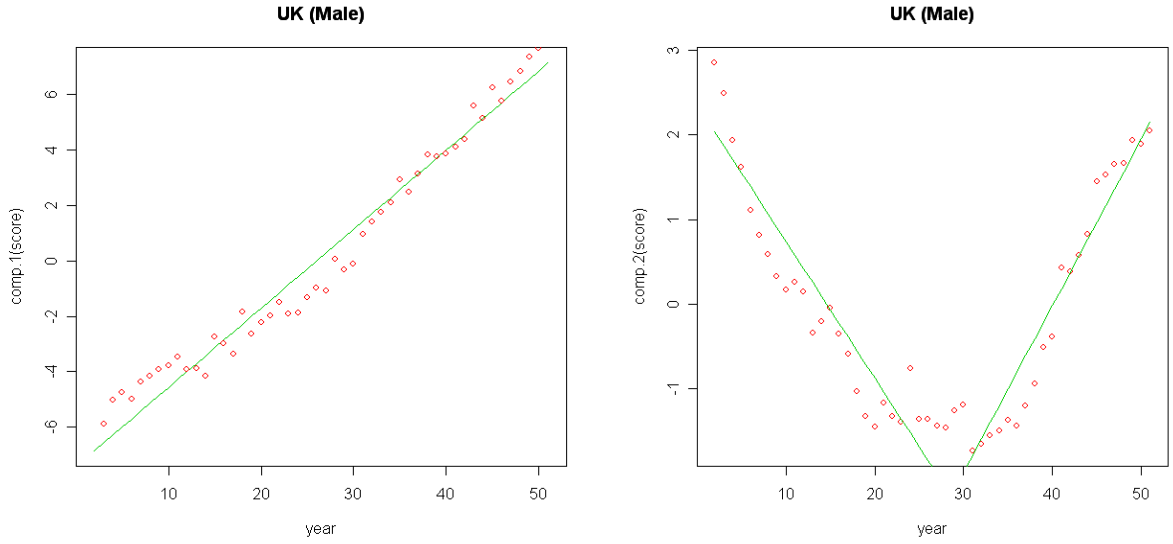



Figure 5. First and Second PC of British Men

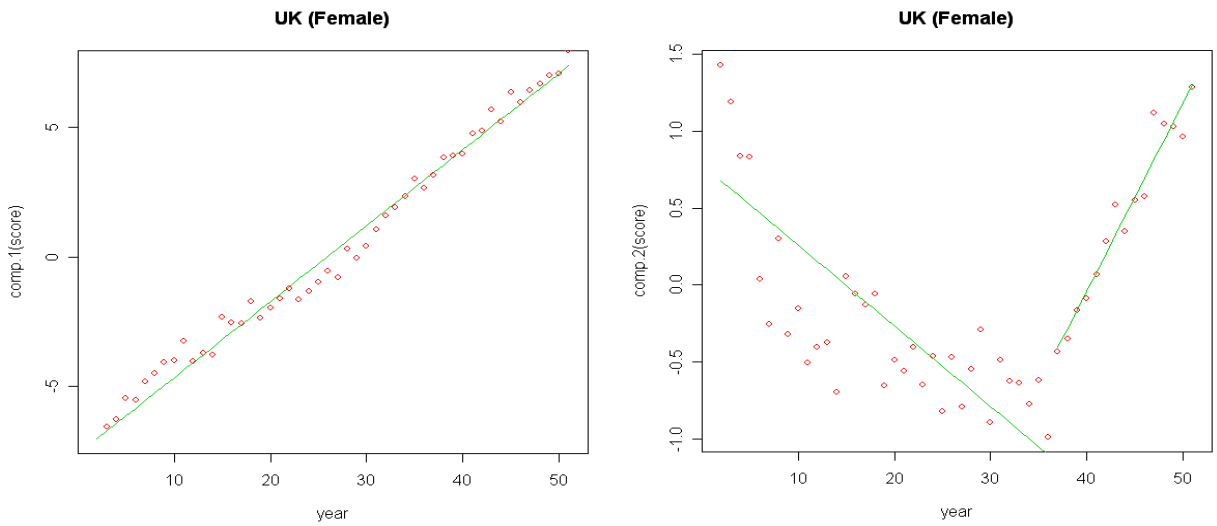



Figure 5(cont.). First and Second PC of British Women

As expected, the first PC of all illustrated countries for both men and women is very close to a straight line of time, just as in the LC model. In contrast, the second PC is not a straight line; though it still looks somewhat like a linear function of time, it behaves quite differently before and after a certain cutoff point, such that the slopes before and after the cutoff point might not be the same. The slopes before and after the cutoff point also vary with countries and genders. For example, in Figure 5, the slope of British women after the cutoff point is steeper than that before the point, whereas the slope (in absolute value) for British men looks similar both before and after the cutoff point. In all cases, the second PC has a different sign before and after the cutoff point.

### 3.2. The PCA Model

Because the second PC reveals different behaviors before and after the cutoff point, we introduce an indicator function to modify the original LC model. The idea of using an indicator function is similar to that of the spline function; it can help reduce the number of parameters in the model. Adding the second PC to the LC model, we derive the proposed model:

$$\ln\left(\frac{m_{x,t}}{m_{x,0}}\right) = \beta_x \kappa_t + \beta_x^* \kappa_t^*$$
$$\equiv \beta_x (a + bt) + \beta_x^* \{(a_1 + b_1 t\, I[t < t_0]) + (a_2 + b_2 t\, I[t \geq t_0])\}, \qquad (3.1)$$
$$\equiv \beta_x (a + bt) + \beta_x^* (a_1^* + b_1^* t\, I[t < t_0] + b_2^* t\, I[t \geq t_0])$$

where $t_0$ is the cutoff point.

The idea of adding $\beta_x^* \kappa_t^*$ in Equation (3.1) is similar to adding $\kappa_t(j)$ in Equation (2.2) and $\kappa_c^*$ in Equation (2.3). The parameters $a, b, a_1^*, b_1^*, b_2^*$, and $t_0$, can be estimated using the least squares (like ordinary regression), after we find the first two PCs from the PCA (see C). In the two-PC model, the number of parameters used is approximately 50 percent more than that of the original LC model. If all the components of $\kappa_t(j)$ in Equation (2.2) and $\kappa_c^*$ in Equation (2.3) are linear functions of the time or cohort, both equations can be simplified to Equation (2.1). In other words, the parameters $\kappa_t(j)$ and $\kappa_c^*$ cannot be simply linear functions of time, or they could not be used to describe the age shift in the mortality reduction. We use empirical data to determine the possible forms for $\kappa_t(j)$ and $\kappa_c^*$.

Note that the idea of introducing a cutoff point in $\kappa_x^*$ is equivalent to introducing an age shift in the mortality reductions. That is, the mortality reductions before and after the cutoff point differ, and the mortality patterns shift at the cutoff point. The mortality improvements for the elderly groups thus are especially significant after the cutoff point. In all countries, the elderly experience the largest mortality reductions after the cutoff point, whereas the 20–60 age groups experience the smallest reductions. Taiwanese men are the only case in which we find large reductions in both the younger and the elderly groups. In the next section, we use an empirical study to evaluate the performance of our modification of the LC model.

The computation of our approach is fairly straightforward, similar to the PCA that Hyndman and Ullah (2005) use. The number of age shifts is not limited to 1, and we can use an idea similar to

that used in the cubic spline interpolation to find the optimal polynomial between two age shifts (though we prefer using a linear function). Because the results of the empirical analysis all suggest one age shift, we do not discuss the case for two or more age shifts herein. However, we do not rule out the possibility of including more age shifts by modifying Equation (3.1).

## 4. Empirical Study

In this section, we use the empirical data to evaluate the proposed two-PC model, Equation (3.1). Moreover, we investigate the difference between the proposed PCA model and other well-known mortality models, such as the LC model, the APC model (Renshaw and Haberman, 2006), and the CBD model. We examine both in-sample and out-sample data fitting accuracy.

### 4.1. Data Description and Criteria for Model Selection

We examine mortality modeling for six counties: Japan, Taiwan, Great Britain, France, Canada, and the United States. The mortality data appear in the format of five-year age groups, ranging from 0 to 99 year-old, and are separated for men and women. The data are divided into the fitting period (in-sample) and prediction period (out-sample), such that data from 1970 to 2000 represent the fitting period and the rest are the prediction period (Table 1).

Table 1. Prediction Period for Different Countries[8]

| Country | Taiwan | Japan | USA | Canada | UK | France |
|---|---|---|---|---|---|---|
| Prediction Period | 2001~ 2005 | 2001~ 2006 | 2001~ 2004 | 2001~ 2004 | 2001~ 2003 | 2001~ 2005 |

We use two criteria to evaluate the performance, or goodness of fit, of the mortality models. The first criterion is the mean absolute percentage error (MAPE), defined as

[8] Due to the availability of HMD mortality data accessed in 2008, we have different prediction period for different countries.

$$MAPE = \frac{1}{n}\sum_{i=1}^{n}\frac{\left|Y_i - \hat{Y}_i\right|}{Y_i} \times 100\%, \tag{4.1}$$

where $Y_i$ and $\hat{Y}_i$ are the actual values and estimated (or predicted) values of mortality[9], and $n$ is the number of observations.

To avoid the possibility of over-parameterization, we also use the Bayes criterion (BIC), which takes the fitting errors and the number of parameters into account. We define the BIC as

$$BIC = l(\hat{\phi}) - \frac{1}{2} v \log N, \tag{4.2}$$

where $l(\hat{\phi})$ is the maximum-likelihood estimate, $v$ is the number of parameters being estimated, and $N$ is the number of observations.

According to these criteria, the ideal model should have the largest value in BIC but the smallest values in MAPE. We compute the values of both MAPE and BIC for the mortality models and discuss the findings in the following section.

## 4.2. Fitting Accuracy

The proposed PCA model can be computed using the regular PCA procedures, which appear in most statistical software, and the corresponding parameter estimates are in C. For the LC model, singular value decomposition (SVD) and its approximation when there are missing values enable us to reach the parameter estimates. For the APC model, since period and cohort effects together can cause linear dependency, it needs to be handled with extra care. Other than the original iteration estimation in Renshaw and Haberman (2006), namely, APC-M, we also try two other estimation procedures: fitting the period effect first and then the cohort effect, or reverse this fitting order. In the following discussion, we call these two procedures APC1 and APC2, respectively. Regarding

[9] We calculate the estimated mortality rate ($\hat{q}(t,x)$) from each model and compare it with the actual mortality rate($q(t,x)$).

the CBD model, we employ the least squares method to fit the mortality curve and obtain the parameter estimates[10].

In general, the APC models (except the APC2) have the best fitting results (Table 2). The APC-M especially outperforms all models except for the females in Taiwan and Canada and the APC-1 model comes in the second, which indicates that including the cohort represents a feasible choice for modifying the LC model. It seemd that the estimation method can make differences in the APC model, where the APC1 model acheives smaller MAPEs than the APC2 model.

Table 2. MAPE for Various Model Fits: Ages 0–99

| **MAPE** | Taiwan Male | Taiwan Female | Japan Male | Japan Female | USA Male | USA Female |
|---|---|---|---|---|---|---|
| **LC** | 7.73% | 7.96% | 5.48% | 7.34% | 4.09% | 3.26% |
| **APC1** | 6.72% | 7.13% | 4.37% | 5.45% | 3.65% | 2.81% |
| **APC2** | 11.37% | 12.33% | 11.82% | 13.68% | 6.18% | 7.74% |
| **APC-M** | 6.64% | 7.70% | 3.91% | 2.97% | 2.82% | 2.53% |
| **CBD** | 86.80% | 101.59% | 63.98% | 74.53% | 50.33% | 62.47% |
| **PCA** | 7.58% | 8.06% | 4.99% | 5.44% | 5.47% | 3.88% |
| | | | | | | |
| **MAPE** | Canada Male | Canada Female | UK Male | UK Female | France Male | France Female |
| **LC** | 4.88% | 3.97% | 4.65% | 4.44% | 5.47% | 4.89% |
| **APC1** | 4.41% | 3.33% | 4.29% | 4.15% | 5.16% | 4.20% |
| **APC2** | 7.24% | 9.97% | 7.35% | 8.26% | 8.43% | 10.36% |
| **APC-M** | 3.45% | 3.54% | 3.51% | 3.88% | 3.94% | 3.77% |
| **CBD** | 58.01% | 70.10% | 65.60% | 72.03% | 53.56% | 70.10% |
| **PCA** | 5.40% | 3.85% | 4.14% | 4.16% | 5.27% | 4.36% |

The results in Table 2 are based on groups of 0 to 99 year-olds. However, because the CBD model is designed for older age groups, we also consider the case only for older age groups (i.e., 60 to 99 year-olds) as a fair comparison. Table 3 shows the results for these older age groups; the CBD provides much smaller MAPE values. However, for the case of ages 60 to 99, we found that the APC-M model still outperforms the CBD model, and the proposed PCA & APC-1 models are well

[10] The parameter estimation method follows the papers of Wilmoth(1996), Renshaw and Haberman (2006) and Cairns et al. (2006). The corresponding parameter estimates for LC, APC, and CBD models are available from the authors.

comparable with the CBD model. We shall continue the comparisons with respect to the values of BIC to ensure the model is not overfitting.

Table 3. MAPE for Various Model Fits: Ages 60–99[11]

| MAPE | Taiwan Male | Taiwan Female | Japan Male | Japan Female | USA Male | USA Female |
|---|---|---|---|---|---|---|
| **LC** | 7.83% | 8.1% | 5.18% | 8.69% | 2.22% | 2.35% |
| **APC1** | 6.45% | 6.76% | 3.85% | 5.83% | 2.00% | 1.80% |
| **APC-M** | 4.05% | 4.35% | 2.03% | 1.28% | 0.86% | 0.98% |
| **CBD** | 5.28% | 6.25% | 3.85% | 4.52% | 1.85% | 3.74% |
| **PCA** | 7.09% | 5.35% | 4.07% | 3.86% | 3.92% | 2.25% |
| | | | | | | |
| **MAPE** | **Canada Male** | **Canada Female** | **UK Male** | **UK Female** | **France Male** | **France Female** |
| **LC** | 2.71% | 2.79% | 3.25% | 2.36% | 2.92% | 3.18% |
| **APC1** | 2.23% | 2.31% | 2.88% | 2.30% | 2.74% | 2.84% |
| **APC-M** | 1.34% | 1.08% | 1.19% | 1.18% | 1.51% | 1.20% |
| **CBD** | 1.51% | 2.79% | 1.96% | 2.97% | 3.19% | 4.77% |
| **PCA** | 2.95% | 1.88% | 2.67% | 2.36% | 3.52% | 2.55% |

We further compare the models according to the BIC criterion for the cases of ages 0-99 and 60-99 (Tables 4 and 5). It should be noted that a larger BIC value indicates a better fit. The PCA-M still achieves the best fit, but unlike the MAPE results, the PCA-M model achieves better fits than the APC1 model. For ages 0-99, the proposed PCA model outperforms the LC and CBD models except for Taiwan males and USA males. For higher ages of 60-99, the proposed PCA model outperforms the LC model except for Taiwan males and the CBD model except for Taiwan females, USA male, canada male and UK male.

Table 4. BIC for Various Model Fits: Ages 0–99

| BIC | Taiwan Male | Taiwan Female | Japan Male | Japan Female | USA Male | USA Female |
|---|---|---|---|---|---|---|
| **LC** | -25301.0 | -20533.7 | -12509.6 | -20648.9 | -6778.8 | -6094.6 |

[11] Since APC2 does not perform well, we shall disregard it from the following analysis on age effect and mortality forecasting.

| | | | | | | |
|---|---|---|---|---|---|---|
| **APC1** | -21699.6 | -15199.8 | -9845.5 | -12726.8 | -6800.1 | -5946.2 |
| **APC2** | -28035.0 | -25763.2 | -32866.5 | -33094.7 | -12237.2 | -17018.1 |
| **APC-M** | -9149.9 | -8932.8 | -6345.8 | -5321.1 | -5590.9 | -5207.8 |
| **CBD** | -702657.0 | -819697 | -581120.0 | -770534.0 | -441254.0 | -558629.0 |
| **PCA** | -26296.7 | -20157.9 | -8487.6 | -7538.3 | -8001.5 | -5610.9 |
| | | | | | | |
| **BIC** | Canada Male | Canada Female | UK Male | UK Female | France Male | France Female |
| **LC** | -8209.4 | -7001.3 | -7507.6 | -6344.1 | -8372.1 | -7323.9 |
| **APC1** | -7890.7 | -6921.8 | -7355.8 | -6498.6 | -8382.6 | -7243.7 |
| **APC2** | -13337.4 | -22134.5 | -14752.8 | -17001.2 | -19752.2 | -22805.5 |
| **APC-M** | -2914.2 | -2676.0 | -2885.8 | -2666.0 | -3028.1 | -5219.7 |
| **CBD** | -535945.0 | -660736.0 | -512574.0 | -599806.0 | -529171.0 | -789799.0 |
| **PCA** | -7162.6 | -5501.3 | -6649.4 | -5911.4 | -7856.3 | -5935.4 |

Table 5. BIC for Various Model Fits: Ages 60–99

| **BIC** | Taiwan Male | Taiwan Female | Japan Male | Japan Female | USA Male | USA Female |
|---|---|---|---|---|---|---|
| **LC** | -19585.1 | -15637.9 | -9061.8 | -17518.2 | -3370.8 | -3465.0 |
| **APC1** | -14811.3 | -9299.4 | -6462.1 | -9628.6 | -3409.9 | -3205.6 |
| **APC-M** | -4476.0 | -4554.3 | -3386.2 | -2746.0 | -2699.0 | -2624.8 |
| **CBD** | -7625.0 | -8656.0 | -6177.0 | -6816.0 | -3469.0 | -4708.0 |
| **PCA** | -5666.1 | -12408.0 | -5485.5 | -4683.7 | -4530.5 | -3014.4 |
| | | | | | | |
| **BIC** | Canada Male | Canada Female | UK Male | UK Female | France Male | France Female |
| **LC** | -4060.1 | -4104.5 | -4604.2 | -3515.7 | -4279.8 | -4483.7 |
| **APC1** | -3965.1 | -3864.5 | -4397.5 | -3636.7 | -4308.2 | -4323.1 |
| **APC-M** | -2914.2 | -2676.0 | -2885.8 | -2666.0 | -3028.1 | -2649.6 |
| **CBD** | -3237.0 | -3639.0 | -3379.0 | -4670.0 | -5027.0 | -6505.0 |
| **PCA** | -3842.0 | -3062.3 | -3863.7 | -3453.3 | -4619.1 | -3454.3 |

4.3. Forecasting Accuracy

We use the data from 1970 to 2000 to find parameter estimates for all models, and then derive the mortality predictions on these estimates. Following the literature, we consider the time series of ARIMA processes in mortality forecasts to capture the uncertainty of future mortality. The ARIMA(p,q,r) processes for the proposed PCA model and other three illustrated models are

investigated[12].

The prediction errors (MAPE) of all models based on ages 0–99 appear in Table 6. The PCA model reaches the smallest MAPE and represents the best performance in terms of mortality projections for all illustrated countries. The results are similar if the comparisons are based on ages 60–99 (see Table 7) but the CBD perform the best in most countries. Also, in all cases, the APC-M model has the smallest fitting errors but has the largest prediction errors. On the other hand, the fitting and prediction errors of APC1 model are more stable. It seems that the model performance of the APC model can be highly influenced by the estimation method and it requires a further study. In addition, Dowd et al.(2008) point out the problem of parameter uncertainty on APC model. They conduct a back-testing framework and find that the APC mortality model repeatedly shows evidence of considerable instability if the parameter uncertainty is taken into account.

Although previous work shows that adding the cohort effect can improve model fit, our empirical results suggest another possibility. That is, the age-shift modification may be comparable to adding the cohort effect, without sacrificing the ease of computation advantage of the original LC model.

Table 6. MAPE for Various Models, Forecasting: Ages 0–99

| **MAPE** | Taiwan Male | Taiwan Female | Japan Male | Japan Female | USA Male | USA Female |
|---|---|---|---|---|---|---|
| **LC** | 14.47% | 14.27% | 13.06% | 17.73% | 8.17% | 6.40% |
| **APC1** | 15.81% | 14.61% | 15.57% | 25.16% | 10.80% | 6.16% |
| **APC-M** | 27.66% | 36.68% | 44.29% | 24.72% | 7.37% | 11.36% |
| **CBD** | 81.81% | 99.89% | 47.03% | 53.66% | 45.84% | 51.34% |
| **PCA** | 10.82% | 14.13% | 7.72% | 11.42% | 6.54% | 5.66% |
| | | | | | | |
| **MAPE** | Canada Male | Canada Female | UK Male | UK Female | France Male | France Female |
| **LC** | 10.81% | 7.76% | 8.93% | 7.35% | 12.92% | 10.73% |
| **APC1** | 14.89% | 7.02% | 11.97% | 7.62% | 15.31% | 9.57% |
| **APC-M** | 12.75% | 13.14% | 11.38% | 18.49% | 11.96% | 25.79% |

[12] The corresponding ARIMA(p,q,r) process for different countries and models are available from the authors.

| | | | | | | |
|---|---|---|---|---|---|---|
| **CBD** | 53.30% | 60.62% | 54.67% | 60.30% | 47.55% | 59.62% |
| **PCA** | 9.82% | 7.77% | 5.96% | 6.73% | 11.70% | 10.10% |

Table 7. MAPE for Various Models, Forecasting: Ages 60–99

| **MAPE** | Taiwan Male | Taiwan Female | Japan Male | Japan Female | USA Male | USA Female |
|---|---|---|---|---|---|---|
| **LC** | 8.45% | 17.54% | 15.63% | 30.59% | 7.03% | 3.37% |
| **APC1** | 9.43% | 7.07% | 5.77% | 11.5% | 10.51% | 4.55% |
| **APC-M** | 36.43% | 67.81% | 38.51% | 13.8% | 7.01% | 18.06% |
| **CBD** | 3.10% | 3.41% | 2.89% | 5.49% | 5.52% | 5.01% |
| **PCA** | 7.32% | 6.19% | 3.61% | 4.57% | 8.64% | 3.69% |
| | | | | | | |
| **MAPE** | Canada Male | Canada Female | UK Male | UK Female | France Male | France Female |
| **LC** | 9.48% | 3.42% | 10.88% | 6.63% | 5.89% | 5.57% |
| **APC1** | 13.30% | 2.49% | 15.67% | 6.96% | 9.53% | 4.26% |
| **APC-M** | 20.86% | 13.04% | 10.81% | 21.25% | 6.78% | 31.03% |
| **CBD** | 4.23% | 4.99% | 3.38% | 3.03% | 6.63% | 8.43% |
| **PCA** | 7.13% | 3.86% | 5.94% | 5.22% | 4.89% | 4.80% |

Inspecting the forecasting errors further, we find that the errors of all models are smaller for women, except in the case of Japan. Thus, the mortality rates of men are likely to experience bigger fluctuations. On average, the forecasting errors associated with Asian countries (Taiwan and Japan) and France are the largest, whereas those of the U.S. and U.K. data are the smallest. Judging from the increments of life expectancy in the past (Figure 1), we hypothesize that larger increments might induce larger forecasting errors; Taiwan, Japan, and France have the largest increases in life expectancies.

## 5. Application

Many researchers have demonstrated that the use of a stochastic mortality model can help measure longevity risk for annuity providers and pension plans and thus that it is an useful tool to

handle problems associated with mortality improvements (Cairns et al., 2006; Antolin, 2007; Bauer and Weber, 2007; Hari et al., 2008 ). In this section, we illustrate how we use the proposed PCA model to measure the influence of mortality improvements on life annuity products. Moreover, we investigate how predicted mortality might affect the price of an annuity if we use different mortality models. In particular, we compare the proposed PCA model and the LC model, using as comparison products both immediate and deferred whole life annuity, at various issuing ages, with a limiting age of 100 years.

### 5.1. Measuring the Effect of Longevity Risk on Annuity Pricing

To measure the effect of longevity risk on annuity pricing, we assume that actuaries use the period mortality table[13] to price life annuities, then compare the annuity price with that using the dynamic mortality rates according to the proposed PCA model. Because of mortality improvements, the annuity price calculated with period mortality tables should be underestimated when comparing with that calculated using the dynamic mortality rates. Thus, we analyze the underestimated ratios of annuity price for both immediate whole life and deferred whole life annuities at different issuing ages in Figures 6 and 7, respectively.

As expected, using the period mortality tables underestimates the annuity prices. The ratios underestimated differ, depending on the issuing age, gender, countries, and types of annuity product. For immediate annuity products in general, the underestimate ratio is more noticeable for women in Taiwan, Japan, and France but more significant for men in the United States, Canada, and United Kingdom (see Figure 6 and Figure D-1(Appendix D)). Moreover, longevity risk is more significant for women in Asia (Taiwan and Japan). The underestimating ratio also is greater for higher issue ages, such as between ages 40 and 80 years. Of all combinations, Japanese women have the biggest underestimate at issuing age 80, at more than 9 percent, whereas it is less significant for Canada and the United States. Note that the ratio of the underestimate is not monotonic with the issuing age,

[13] We use the last-year mortality experience as the period mortality table.

partly because of the limiting age of 100 years for the annuity products.

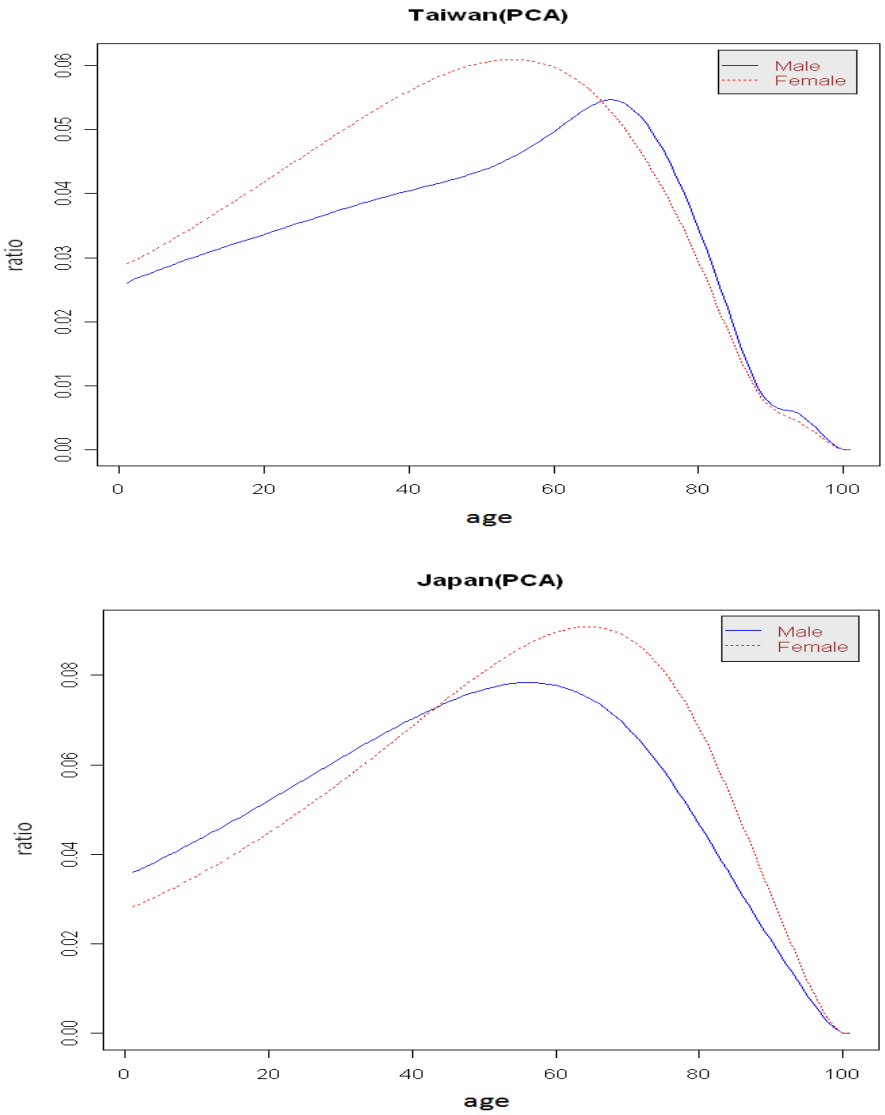



Figure 6. Underestimated Ratios for Immediate Life Annuities, PCA Model (Taiwan and Japan)

For deferred life annuity products, both the issuing age and the deferred period have an impact on the annuity price. We list the contour plots for the ratio underestimated as a function of the issuing age and the deferred period together in Figure 7 and Figure D-2(Appendix D). To demonstrate how we read the outputs, consider a 20-year deferred whole life annuity, with an issuing age of 40 years, as an example. For that product, the underestimate for women is more significant in Taiwan and less so in North America and Europe. Specifically, the underestimate for men is approximately 10–19 percent, and that for women is around 7–16 percent. Of all

combinations, Japanese men experience the greatest ratio of underestimation in general.

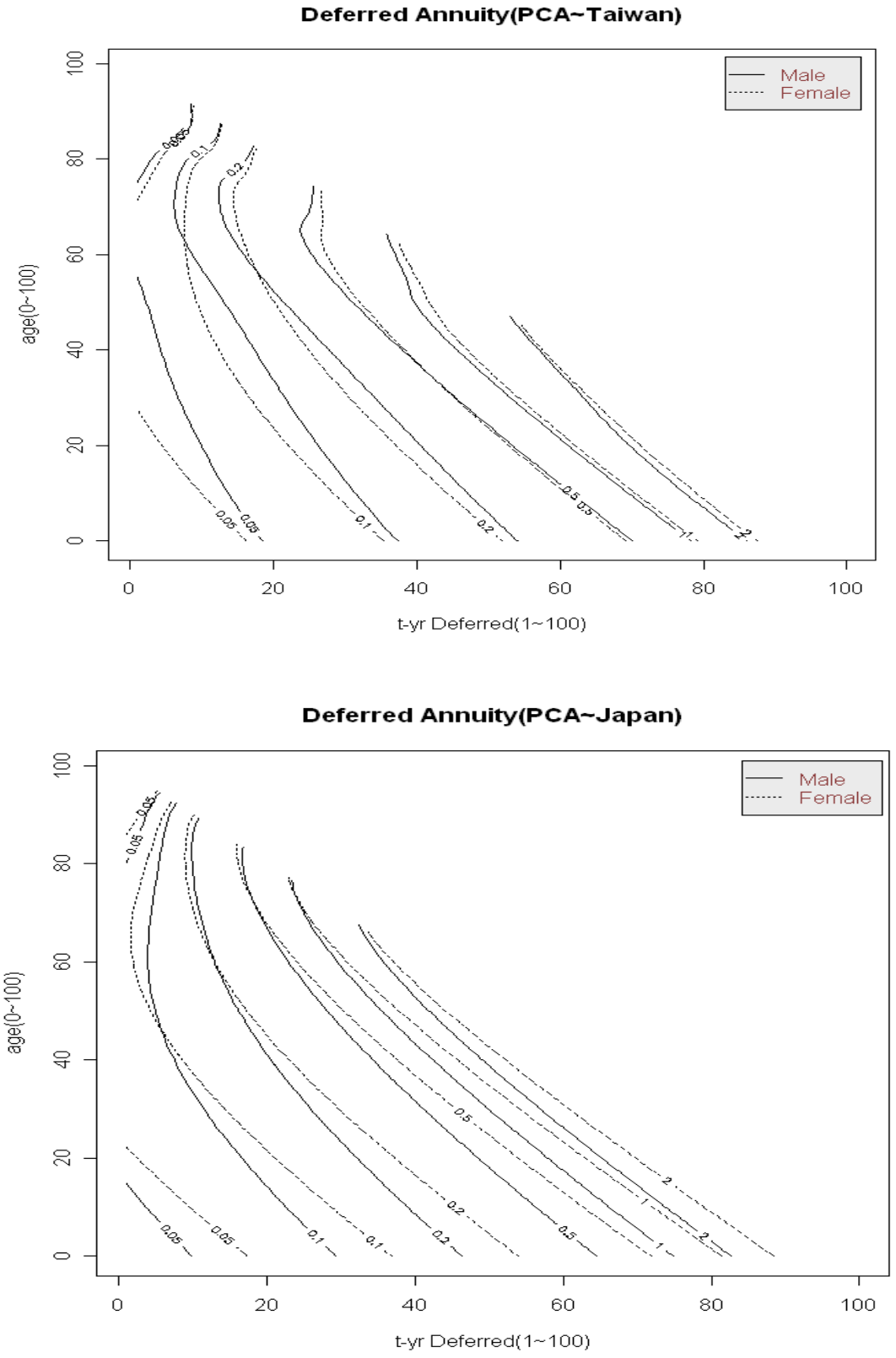
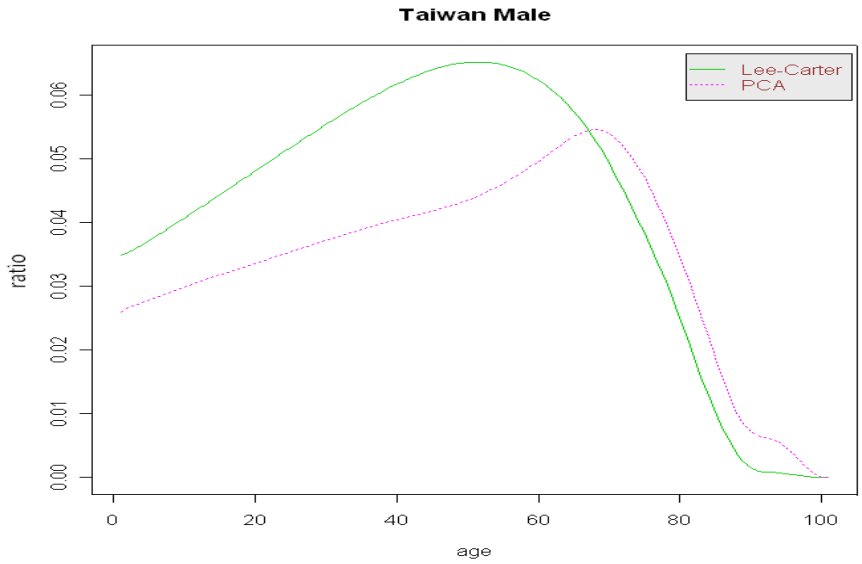



Figure 7. Underestimated Ratios for Deferred Life Annuities, PCA Model

5.2. Analysis of Model Risk

Using a proper mortality model for pricing life annuities is essential. In the following, we address model risk by calculating the value of immediate life annuity obtained from different mortality models. For a comparison purpose, we illustrate it with the proposed PCA model and the LC model. The prices for immediate life annuities for Taiwan males and Japan males, according to these two models, appear in Figures 8 and 9. The underestimated ratio varies according to the combinations of the mortality model and the country (See Appendix E for other countries). For Britain, Japan, and France, the PCA model produces larger underestimates, but the LC model results in a larger underestimate for the United States. These findings are consistent for both men and women. However, for Taiwan and Canada, neither model dominates. In Taiwan, using the LC model results in a larger underestimate for men but not for women, whereas the case of Canada shows the opposite results. Furthermore, the issuing age may have some influence on the underestimation. For example, in Taiwan, Britain, and France, the younger and older age groups reveal different patterns in the ratio of underestimation.



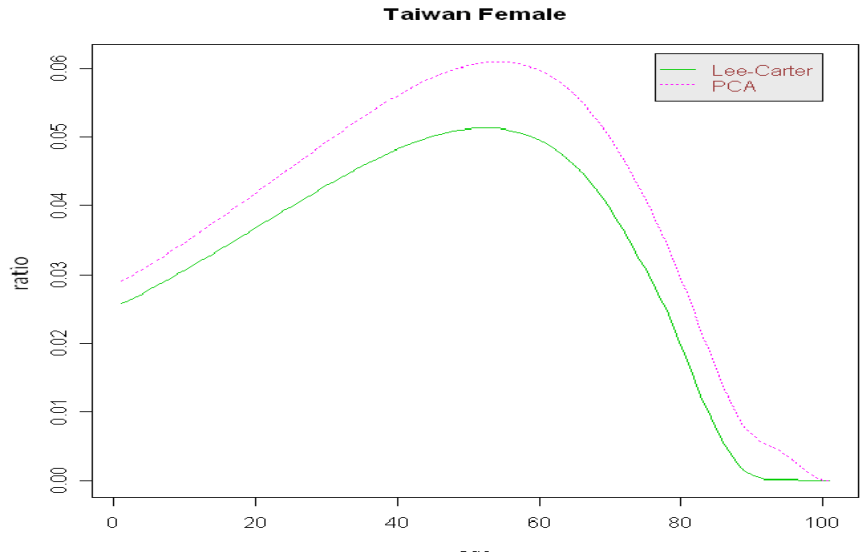



Figure 8. Differences of Models in Immediate Whole Life Annuities: Taiwan

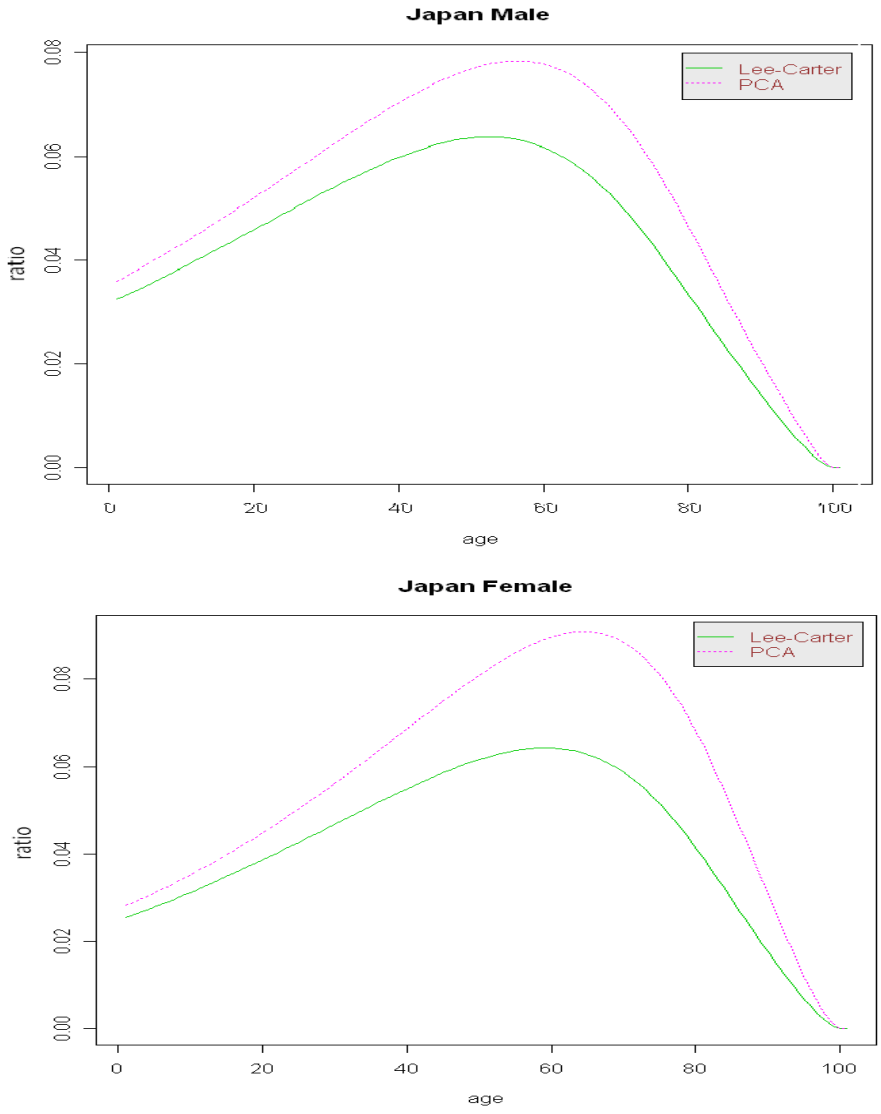



Figure 9. Differences of Models in Immediate Whole Life Annuities: Japan

## 6. Conclusions and Discussion

The Lee-Carter model has received significant attention for its ability to model and project mortality rates since 1992. Because its computation is fairly straightforward and it achieves strong accuracy in its predictions, the LC model probably is the most popular approach for population projections. However, there are two limitations restricting the LC model in pricing the longevity risk: the parameters are assumed to be constant over time and the limiting mortality is zero for all ages. These limitations have prompted a lot of discussions and triggered many modifications. The most recent development in modifying the LC model incorporates the cohort effect, such as in the cohort model (APC model) proposed by Renshaw and Haberman (2006). However, introducing the cohort effect into the model might create problems in parameter estimation, a common problem of the APC model, which also occurs in our empirical study. This modification therefore must be handled with care. Still, because the APC model has the advantage of easy interpretation, we think that the stability of estimation method and its forecasting accuracy is also worth further exploration.

Instead of incorporating the cohort effect, we propose an alternative modification in this study, designed to deal with the age shifts in mortality reductions. The proposed age shift model uses principal component analysis and significantly improves model fit compared with the LC model and is at least comparable to the APC model and the CBD model. Using mortality data from Great Britain, France, Japan, Taiwan, Canada, and the United States, we find that the proposed method outperforms the other three models: It achieves compatible fitting performances according to the BIC, and the best predicting MAPE in mortality projections. Similar to the LC model, the proposed model is easy to implement, and the linearity in PCs make the prediction fairly straightforward.

We also use the proposed model to illustrate the effect of longevity risk on the price of life annuity products and discuss model risk. In general, the annuity price is significantly underestimated when actuaries ignore the future dynamic when pricing annuities. The effect is more

serious for women in Asia (Taiwan and Japan). Also, future mortality improvements according to the LC model are more conservative than those predicted by the proposed model in Japan, the United Kingdom, and France. In other words, the calculated price of these annuity products would undergo a significant increase for these countries if the proposed method, instead of the LC model, were applied. Other than annuity products, our model can be extended to other applications as well, such as the securitization and hedging of longevity risk.

However, there are also limitations in applying the proposed model. Note that the age-related slope parameters $\beta_x$ and $\beta_x^*$ each are constant in the proposed model, but combining $\beta_x$ and $\beta_x^*$ together gives non-constant effects in different periods, which are different from that in the LC model. But, without introducing new PC's in the two-PC model, the age-related slopes of the two PCs will be fixed after the cutoff point. In the future, we will continue investigating the methodology and criterion for introducing a new PC. The discussion can follow the idea in regression analysis, i.e., the variable selection (e.g., number of PCs or number of age shifts) and the time of intervention (i.e., optimal locations for age shifts).

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## Appendix A. Principal Component Analysis

Principal component analysis (PCA) is a popular method for dealing with multivariate data. It can be used for both data reduction and interpretation. As the dimensions of data increase, the difficulty of summarizing these data also increases. By extracting the components that can better describe data properties, PCA provides a means to condense the data. Intuitively, if more components are extracted, the data properties likely can be preserved. However in practice, a few components usually are sufficient to provide a good summary, whereas too many components can induce noise.

The components extracted are linear combinations of original variables. Suppose there are $k$ variables for each observation, namely, $(x_1, x_2, \ldots, x_k)$, and the PCA attempts to find the principal components $(y_1, y_2, \ldots, y_k)$ that satisfy

$$\begin{array}{l} y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k \\ y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k \\ \vdots \quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \quad\quad \vdots \\ y_k = a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kk}x_k \end{array}, \tag{A.1}$$

where the PCs are uncorrelated. The derivation of the PCA uses the covariance or correlation matrix (Johnson and Wichern, 2002) and its eigen-decomposition. The measure of the efficiency of the data reduction by PCA often entails the percentage of variation explained and the proportion of eigenvalues extracted. Although more PCs can increase the percentage of variation explained and the estimation accuracy of mortality rates, there is no guarantee that more PCs lead to better accuracy in the mortality predictions. Similarly, in a regression model, adding insignificant independent variables can increase $R^2$ but also likely distorts (and worsens) model fit.

Suppose, in a study of mortality rates, $k$ groups of age-specific mortality rates are collected each year. Most studies use logarithms of mortality rates instead of mortality rates and choose between one and three PCs. Heligman and Pollard (1980) suggest a data reduction possibility similar to the idea of PCA: They separate human mortality into three periods, or infant and child,

young ages, and adults. Therefore, their classification reflects a 3-PC model. The Lee-Carter model suggests a 1-PC model, in which the logarithms of central death rates of all age groups decrease linearly (Equation 2.1). Our proposed approach offers an example of a 2-PC model, in which the decreasing trends of the logarithm of central death rates vary at two different time periods. The choices of the number of PCs are critical to the efficiency of the data reduction and interpretation.

The expression in Equation (2.2),

$$\ln(m_{x,t}) = \alpha_x + \sum_{j=1}^{J} \beta_x(j)\ \kappa_t(j) + \varepsilon_{x,t}\,, \tag{2.2}$$

is a generalization of the PCA in Equation (A.1), where the variables $\kappa_t(j)$ can be treated as the PCs. Of course, PCA is not the only decomposition method; Hyndman and Ullah (2005) suggest a functional approach as a possible alternative. Because the computation and interpretation of the PCA are relatively easy though, it remains one of the most popular methods.

# Appendix B. Principal componentAnalysis of Mortality Experience

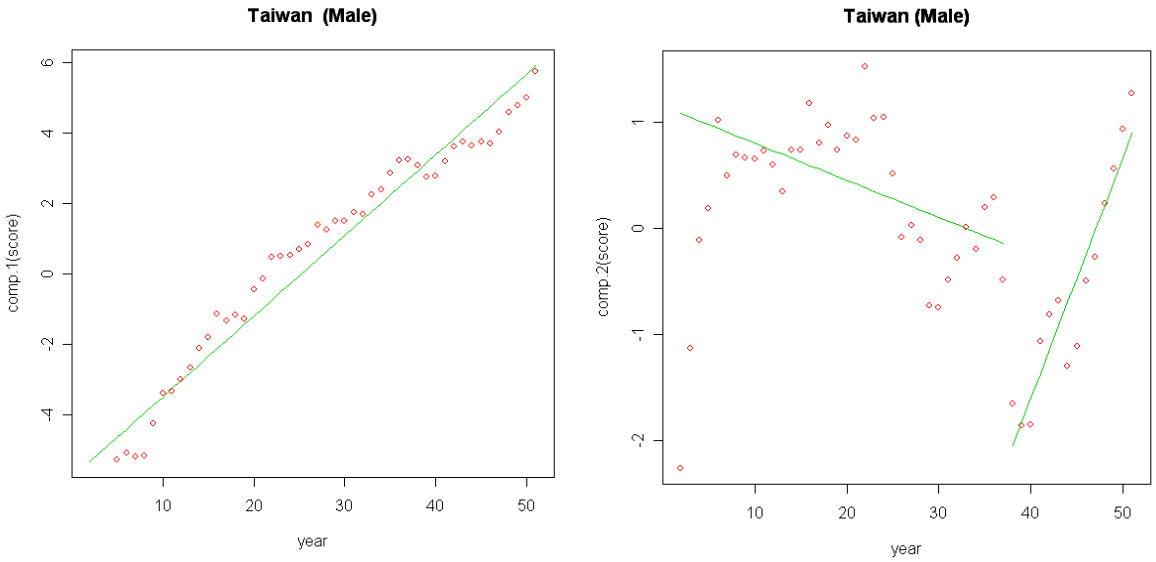


Figure B-1. First and Second PC of Taiwanese Men

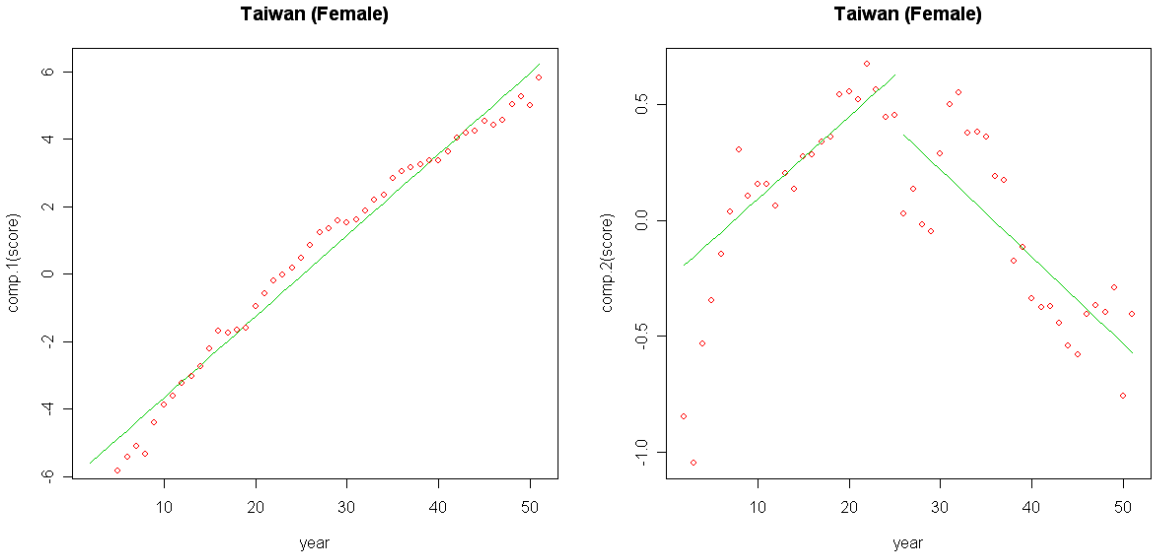


Figure B-2. First and Second PC of Taiwanese Women

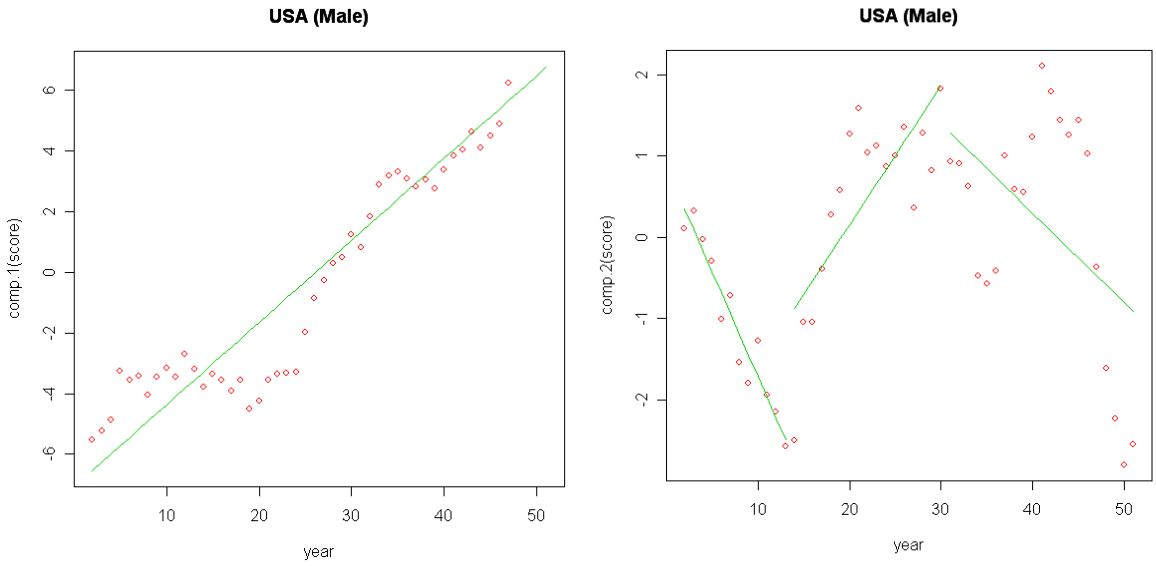


Figure B-3. First and Second PC of American Men

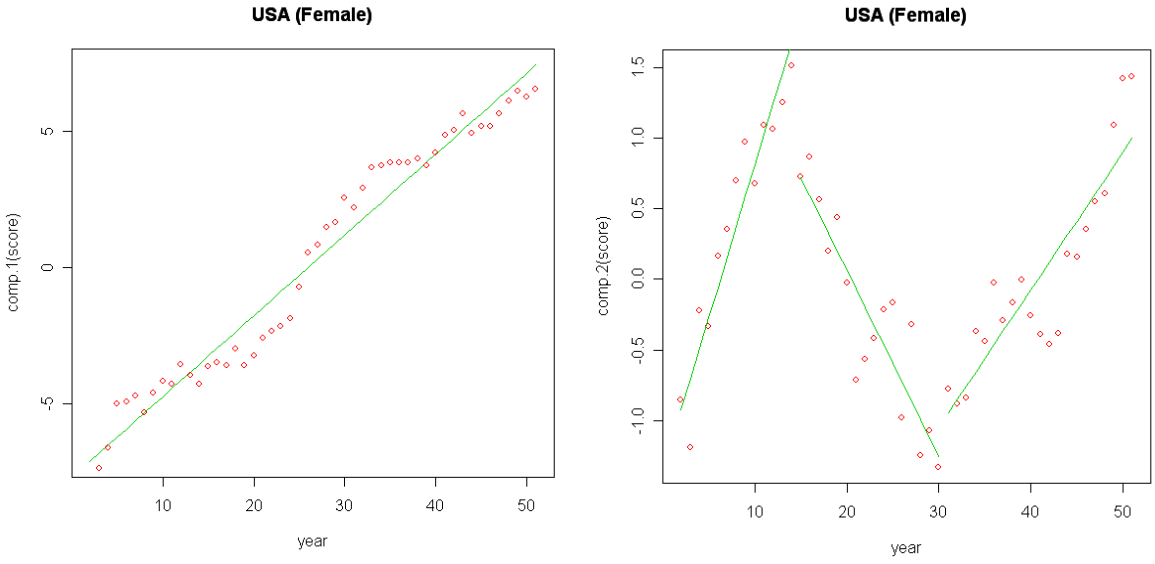


Figure B-4. First and Second PC of American Women

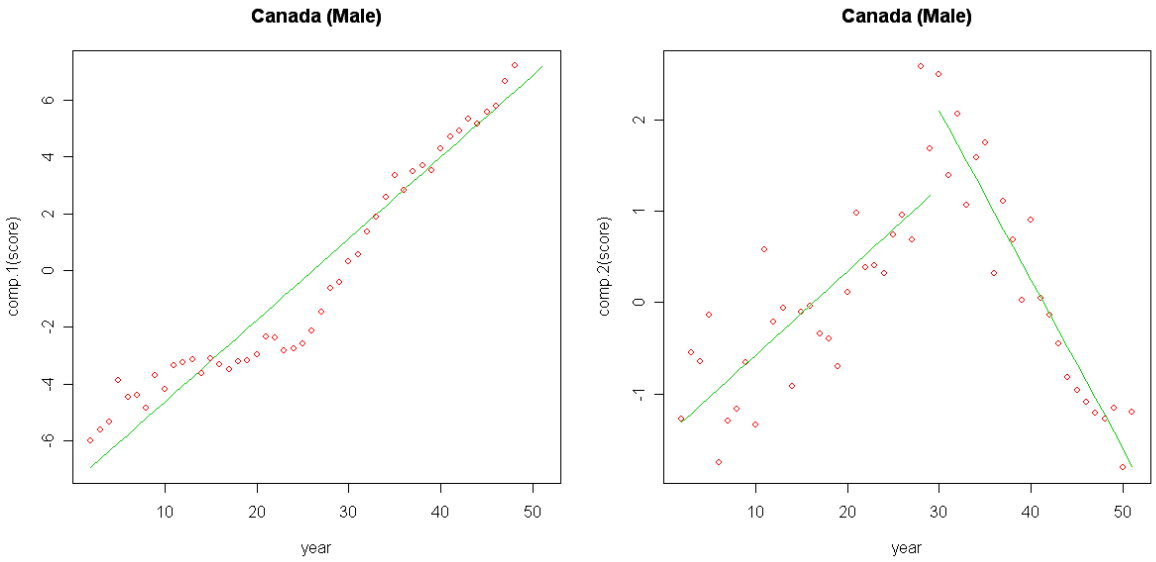


Figure B-5. First and Second PC of Canadian Men

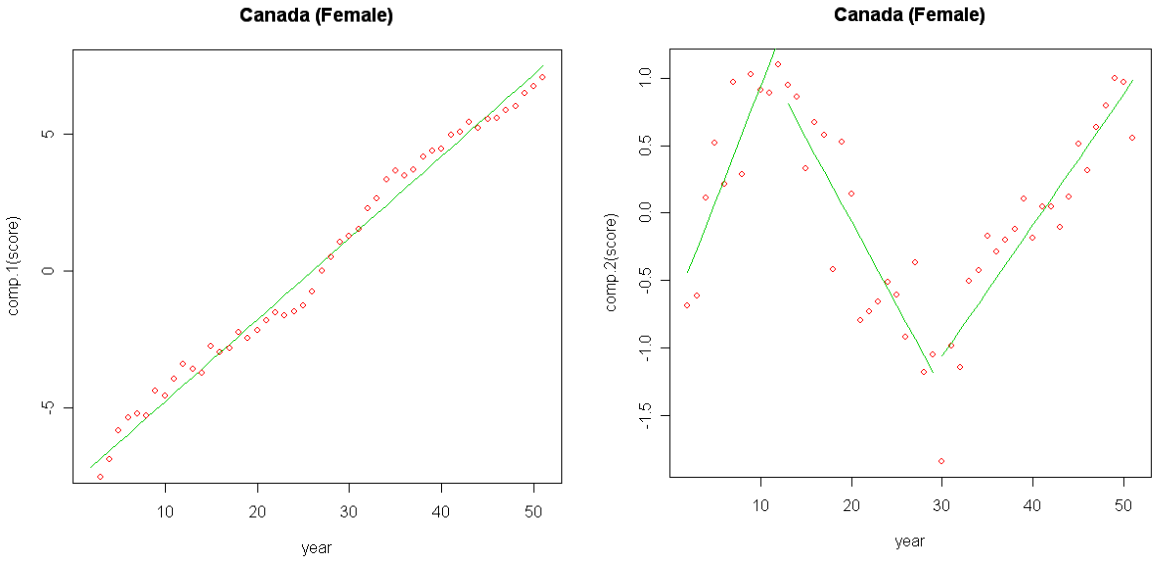


Figure B-6. First and Second PC of Canadian Women

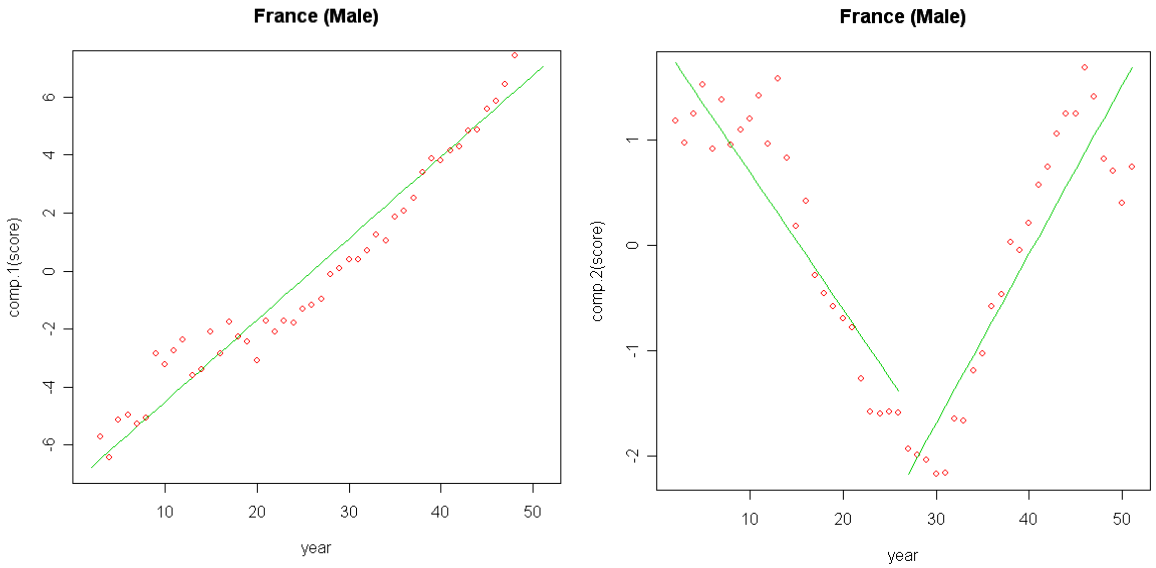


Figure B-7. First and Second PC of French Men

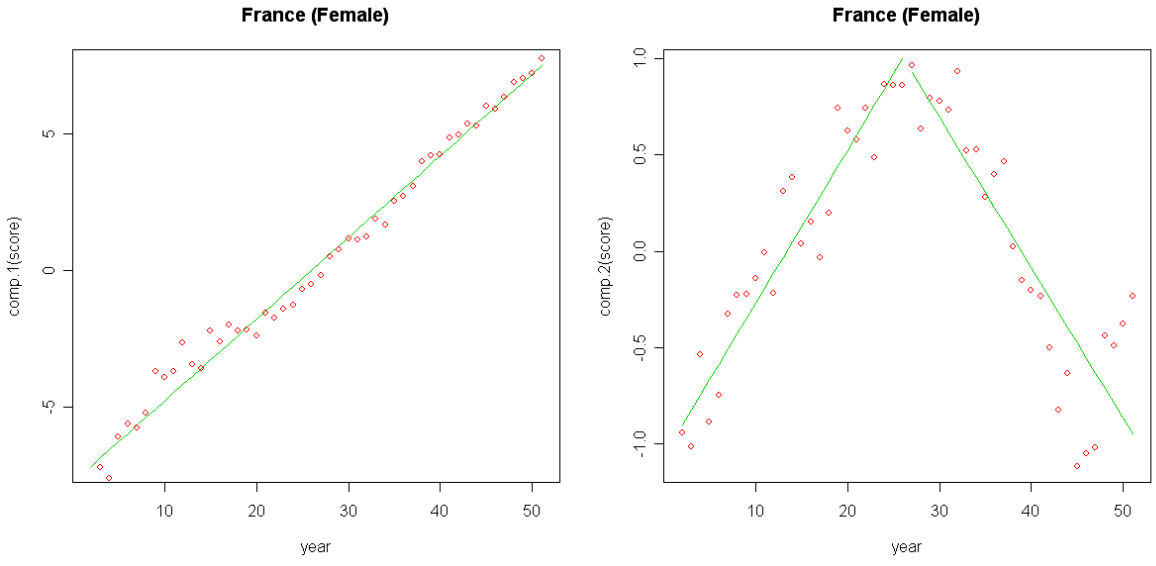


Figure B-8. First and Second PC of French Women

# Appendix C. Parameter Estimates of PCA model

Table C-1. Parameter Estimates of First and Second PC for Taiwanese Men

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -5.785256 | 0.229079 | 0.9488 | 0.9477 |
| Second PC-1 | 1.15156 | -0.03495 | 0.346 | 0.3249 |
| Second PC-2 | -10.64886 | 0.22632 | 0.8779 | 0.8677 |

Table C-2. Parameter Estimates of First and Second PC for Taiwanese Women

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -6.061758 | 0.240597 | 0.9781 | 0.9776 |
| Second PC-1 | -0.263898 | 0.035668 | 0.7689 | 0.7568 |
| Second PC-2 | 1.34786 | -0.03764 | 0.6057 | 0.5893 |

Table C-3. Parameter Estimates of First and Second PC for Japanese Men

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -7.733756 | 0.298617 | 0.988 | 0.9878 |
| Second PC-1 | -1.314420 | 0.107650 | 0.5296 | 0.5072 |
| Second PC-2 | 2.632501 | -0.069705 | 0.8494 | 0.8436 |

Table C-4. Parameter Estimates of First and Second PC for Japanese Women

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -7.739437 | 0.298702 | 0.9912 | 0.991 |
| Second PC-1 | -1.56803 | 0.12414 | 0.7988 | 0.7904 |
| Second PC-2 | 4.654333 | -0.122217 | 0.9444 | 0.9419 |

Table C-5. Parameter Estimates of First and Second PC for American Men

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -7.07006 | 0.27086 | 0.895 | 0.8928 |
| Second PC-1 | 0.86196 | -0.25764 | 0.942 | 0.9367 |

| | | | | |
|---|---|---|---|---|
| Second PC-2 | -3.26547 | 0.17091 | 0.5761 | 0.5478 |
| Second PC-3 | 4.67762 | -0.10964 | 0.222 | 0.1811 |

Table C-6. Parameter Estimates of First and Second PC for American Women

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -7.689643 | 0.296361 | 0.9641 | 0.9633 |
| Second PC-1 | -1.35965 | 0.21704 | 0.926 | 0.9199 |
| Second PC-2 | 2.69923 | -0.13182 | 0.809 | 0.7954 |
| Second PC-3 | -3.96101 | 0.09721 | 0.7891 | 0.778 |

Table C-7. Parameter Estimates of First and Second PC for Canadian Men

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -7.50734 | 0.28771 | 0.937 | 0.9357 |
| Second PC-1 | -1.49487 | 0.09193 | 0.6454 | 0.6322 |
| Second PC-2 | 7.64473 | -0.18498 | 0.9089 | 0.9044 |

Table C-8. Parameter Estimates of First and Second PC for Canadian Women

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -7.744508 | 0.298879 | 0.9852 | 0.9849 |
| Second PC-1 | -0.78546 | 0.17244 | 0.8132 | 0.7945 |
| Second PC-2 | 2.42407 | -0.12435 | 0.776 | 0.7611 |
| Second PC-3 | -3.976459 | 0.097241 | 0.84 | 0.832 |

Table C-9. Parameter Estimates of First and Second PC for British Men

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -7.408933 | 0.285140 | 0.968 | 0.9673 |
| Second PC-1 | 2.35493 | -0.16150 | 0.8608 | 0.8555 |
| Second PC-2 | -7.88350 | 0.19669 | 0.9165 | 0.9125 |

Table C-10. Parameter Estimates of First and Second PC for British Women

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -7.610368 | 0.294015 | 0.9797 | 0.9793 |
| Second PC-1 | 0.783577 | -0.052506 | 0.5435 | 0.5301 |
| Second PC-2 | -4.934575 | 0.122392 | 0.9472 | 0.9432 |

Table C-11. Parameter Estimates of First and Second PC for French Men

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -7.318067 | 0.281331 | 0.9513 | 0.9503 |
| Second PC-1 | 1.98637 | -0.12972 | 0.7858 | 0.7769 |
| Second PC-2 | -6.50202 | 0.16045 | 0.8289 | 0.8289 |

Table C-12. Parameter Estimates of First and Second PC for French Women

| | $a_i$ | $b_i$ | R-Squared: | Adjusted R-squared: |
|---|---|---|---|---|
| First PC | -7.752124 | 0.298922 | 0.9825 | 0.9821 |
| Second PC-1 | -1.061168 | 0.079204 | 0.9017 | 0.8976 |
| Second PC-2 | 3.033244 | -0.077991 | 0.7602 | 0.7497 |

# Appendix D. The Effect of Longevity Risk on Annuity Pricing

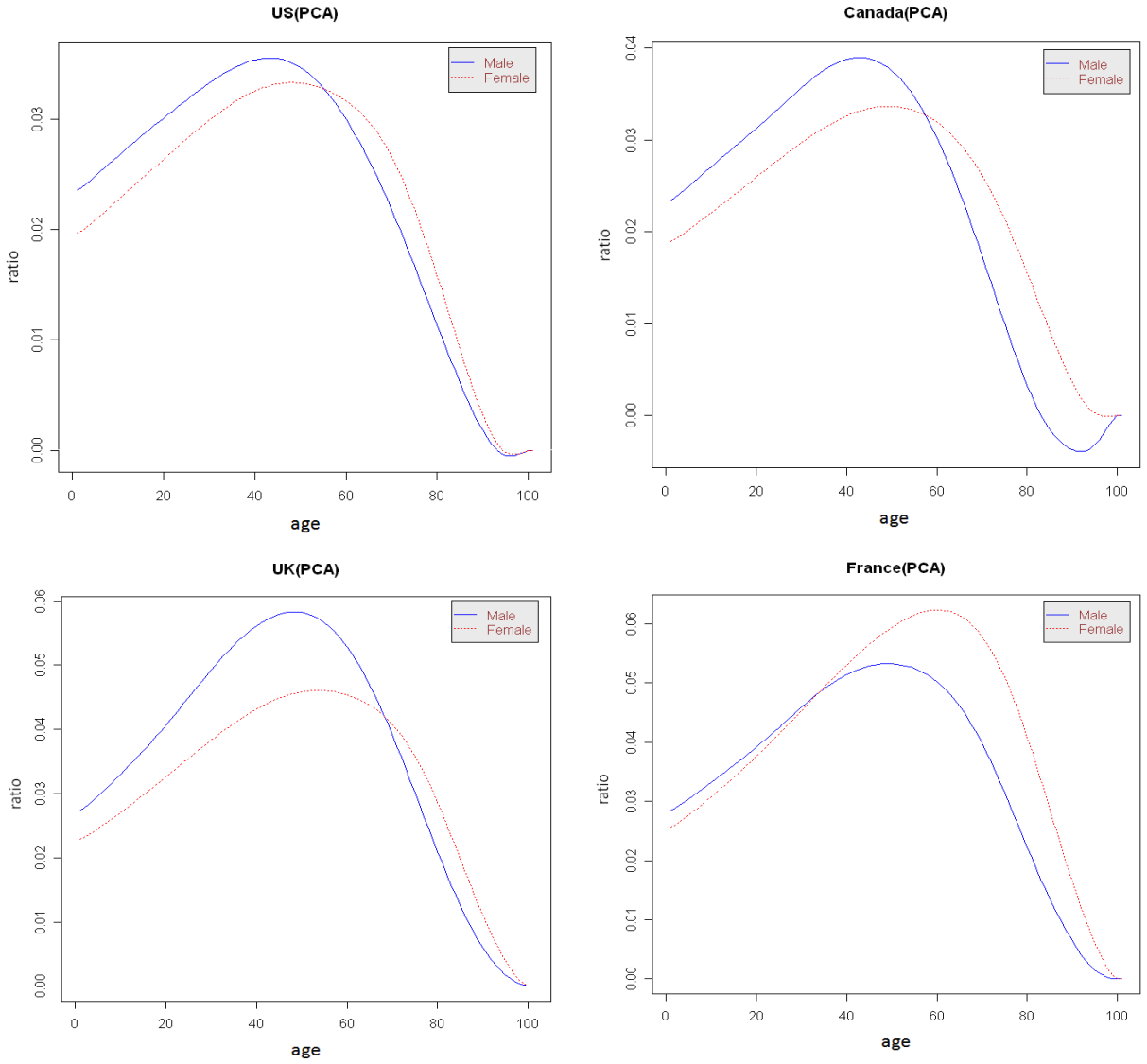


Figure D-1. Underestimated Ratios for Immediate Life Annuities, PCA Model(US, Canada, UK, France)

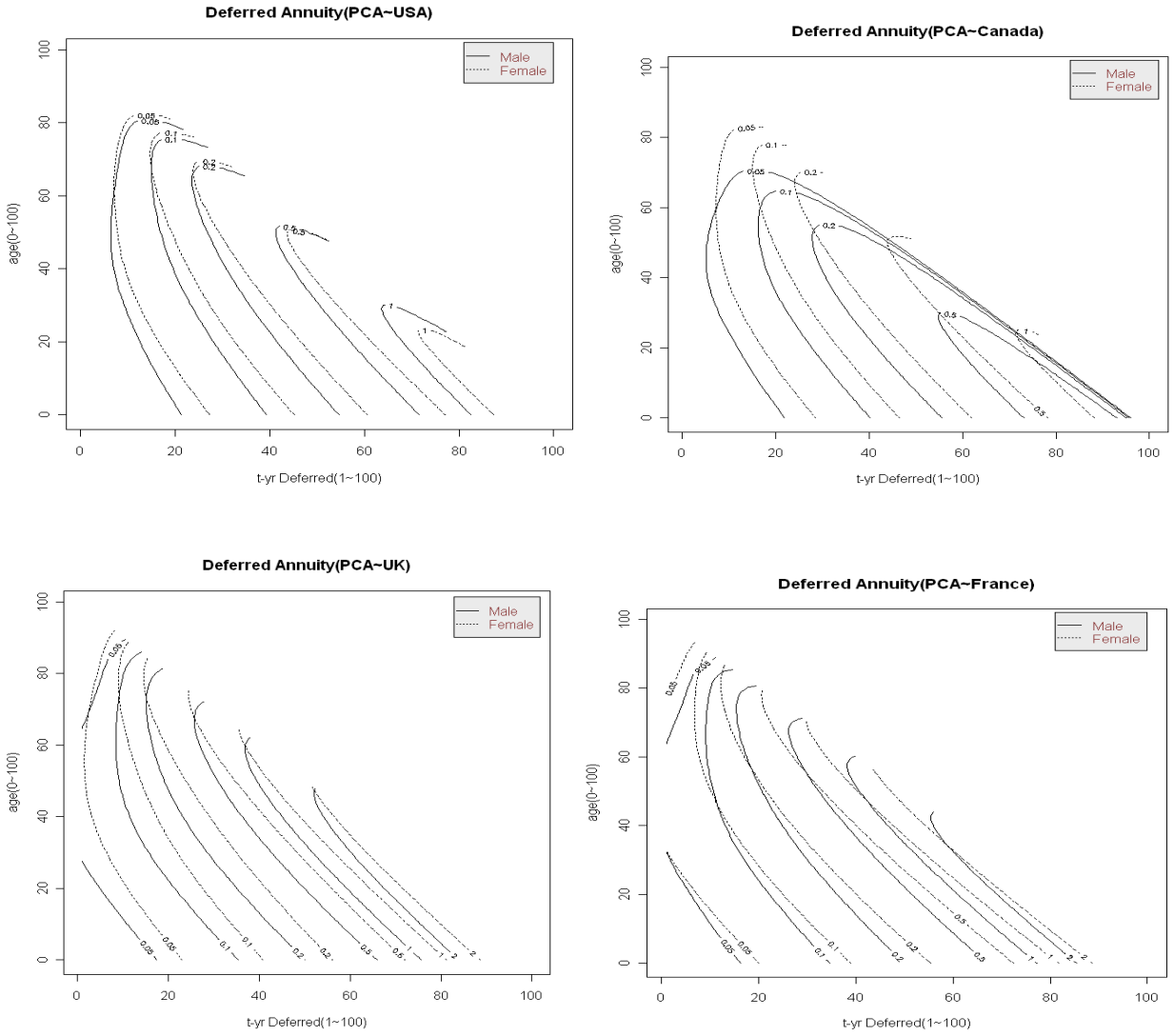


Figure D-2. Underestimated Ratios for Deferred Life Annuities, PCA Model(US, Canada, UK, France)

# Appendix E. Analysis of Model Risk on Annuity Pricing

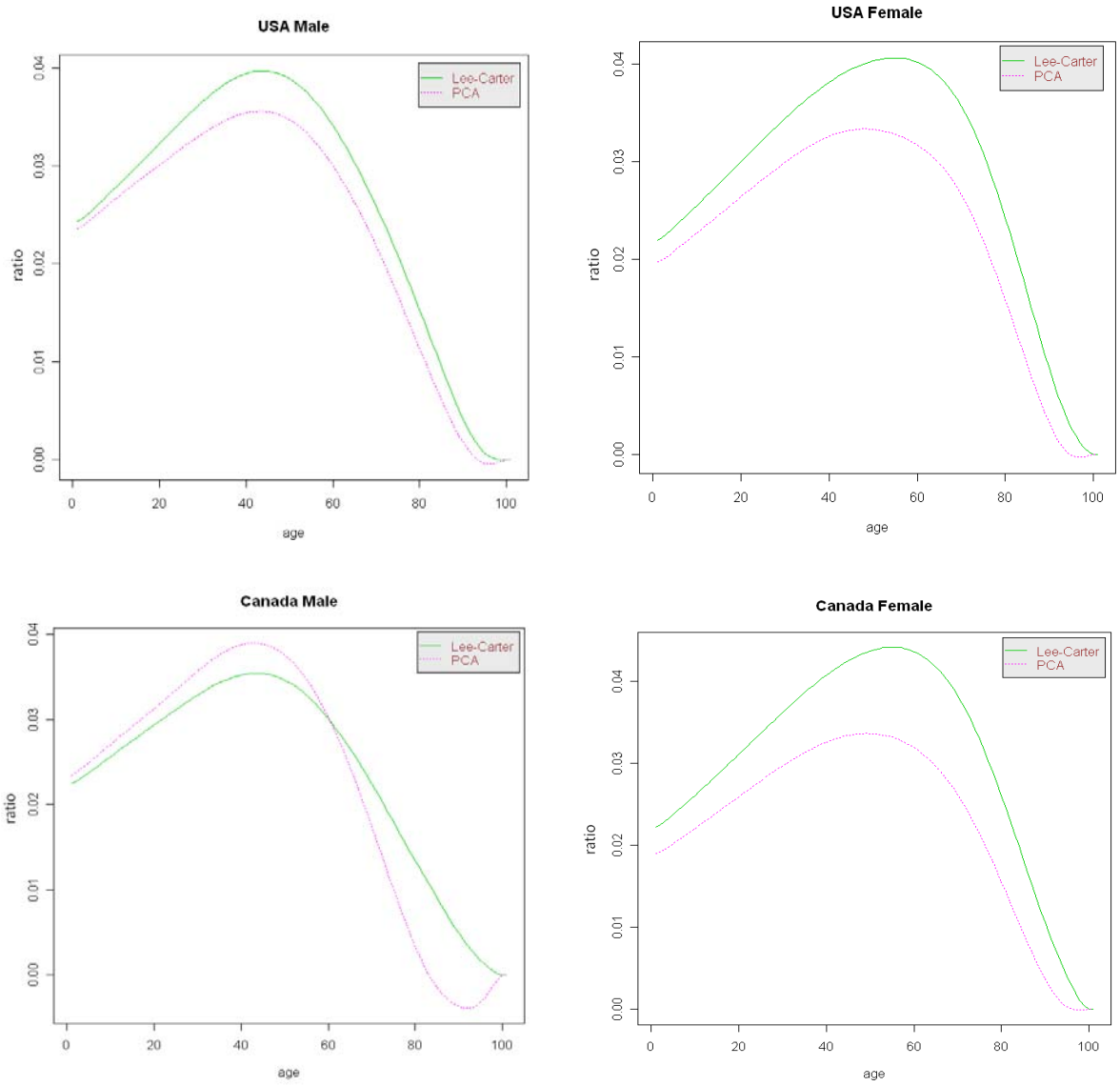


Figure E-1. Differences of Models in Immediate Whole Life Annuities (US, Canada)

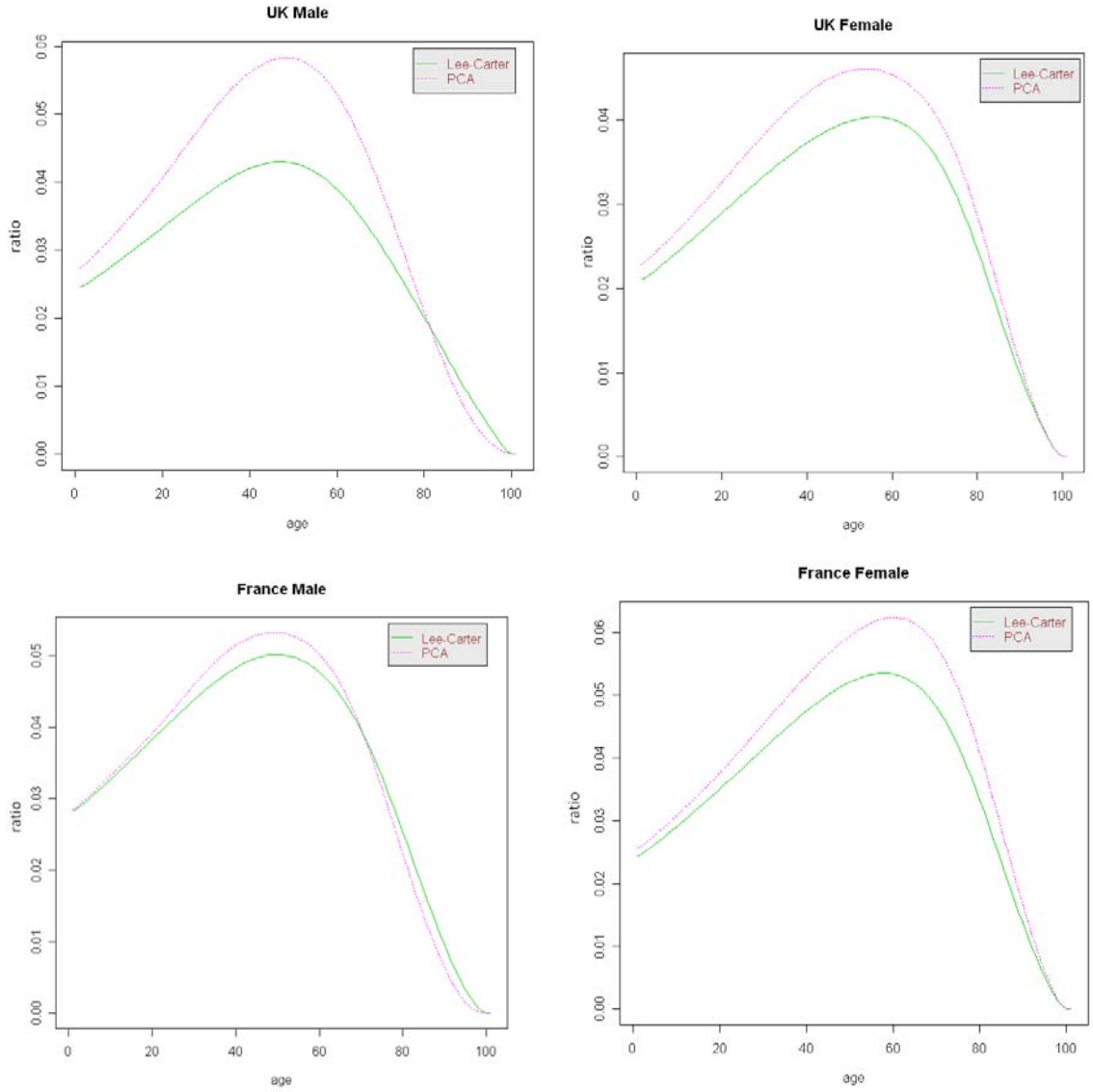


Figure E-2. Differences of Models in Immediate Whole Life Annuities: (U.K, France).